Chapter 6 Written Work

(1) $x=0$ is an ordinary point of a diff eq. Equating coefficents to zero (Homogeneous Eq'n), we obtain the following system. Assume c_0, c_1 are the free variables.

$y=\sum_{n=0}^{\infty} c_n x^n$ is the sol'n

$$\begin{aligned}
2c_2 + 2c_1 + c_0 &= 0\\
6c_3 + 4c_2 + c_1 &= 0\\
12c_4 + 6c_3 + c_2 - \tfrac{1}{3}c_1 &= 0\\
20c_5 + 8c_4 + c_3 - \tfrac{2}{3}c_2 &= 0
\end{aligned}
\qquad
\begin{bmatrix} 0 & 0 & 0 & 2 & 2 & 1 & 0\\ 0 & 0 & 6 & 4 & 1 & 0 & 0\\ 0 & 12 & 6 & 1 & -\tfrac{1}{3} & 0 & 0\\ 20 & 8 & 1 & -\tfrac{2}{3} & 0 & 0 & 0 \end{bmatrix}$$

A: <0, 0, 0, 2, 2, 1, 0; 0, 0, 6, 4, 1, 0, 0; 0, 12, 6, 1, $-\frac{1}{3}$, 0, 0; 20, 8, 1, $-\frac{2}{3}$, 0, 0, 0>

with (LinearAlgebra):

ReducedRowEchelonForm (A)

$$\begin{array}{cccccc|c} c_5 & c_4 & c_3 & c_2 & c_1 & c_0 & \\ \hline 1 & 0 & 0 & 0 & \frac{1}{360} & -\frac{1}{60} & 0\\ 0 & 1 & 0 & 0 & \frac{5}{36} & \frac{1}{8} & 0\\ 0 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{3} & 0\\ 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & 0 \end{array}$$

$c_5 + \frac{1}{360}c_1 - \frac{1}{60}c_0 = 0$

$c_5 = -\frac{1}{360}c_1 + \frac{1}{60}c_0$

$c_4 + \frac{5}{36}c_1 + \frac{1}{8}c_0 = 0$

$c_4 = -\frac{5}{36}c_1 - \frac{1}{8}c_0$

$c_3 - \frac{1}{2}c_1 - \frac{1}{3}c_0 = 0$

$c_3 = \frac{1}{2}c_1 + \frac{1}{3}c_0$

$c_2 + c_1 + \frac{1}{2}c_0 = 0$

$c_2 = -c_1 - \frac{1}{2}c_0$

$c_5 = -\frac{1}{360}c_1 + \frac{1}{60}c_0$

$c_4 = -\frac{5}{36}c_1 - \frac{1}{8}c_0$

$c_3 = \frac{1}{2}c_1 + \frac{1}{3}c_0$

$c_2 = -c_1 - \frac{1}{2}c_0$

This gives

$$\boxed{\begin{aligned} y_1(x) &= c_0\left[1 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{8}x^4 + \tfrac{1}{60}x^5 + \cdots\right]\\ y_2(x) &= c_1\left[x - x^2 + \tfrac{1}{2}x^3 - \tfrac{5}{36}x^4 - \tfrac{1}{360}x^5 + \cdots\right]\\ y &= y_1 + y_2 \end{aligned}}$$

(2) Find the 1^{st} 3 nonzero terms of the power series for $f(x) = \tan(x)$, by using long division of the power series for $\sin(x)$ & $\cos(x)$.

$\sin(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots$

$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$

$$\begin{array}{r|l} & x + \frac{1}{3}x^2 + \frac{2}{15}x^4 \\ \hline 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \\ & -\left(x - \frac{1}{2}x^3 + \frac{1}{24}x^5\right) \\ \hline & \frac{1}{3}x^3 - \frac{1}{30}x^5 \\ & -\left(\frac{1}{3}x^3 - \frac{1}{6}x^5 \cdots\right) \\ \hline & \frac{2}{15}x^5 \end{array}$$

$-\frac{1}{6} + \frac{1}{2} \cdot \frac{3}{3} = \frac{-1+3}{6} = \frac{2}{6} = \frac{1}{3}$

$\frac{1}{120} - \frac{1}{24} \cdot \frac{5}{5} = \frac{-4}{120}$

$= -\frac{1}{30}$

$2\,|\,120$, $2\,|\,60$, $2\,|\,30$, $3\,|\,15$, 5

$2\,|\,24$, $2\,|\,12$, $2\,|\,6$, 3

$-\frac{1}{30} + \frac{1}{6} \cdot \frac{5}{5}$

$= +\frac{4}{30} = +\frac{2}{15}$

$$\boxed{\tan(x) = x + \frac{1}{3}x^2 + \frac{2}{15}x^4 + \cdots}$$

3) Find the 1^{st} 4 nonzero terms of 2 linearly independent solutions of the equation

$$y'' - xy' - y = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$y'' - xy' - y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) c_{n+1} x^{n+1} - \sum_{n=0}^{\infty} c_n x^n$$

$$= 2c_2 + \sum_{n=1}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=1}^{\infty} n c_n x^n - c_0 - \sum_{n=1}^{\infty} c_n x^n$$

$$= 2c_2 - c_0 + \sum_{n=1}^{\infty} \left((n+2)(n+1) c_{n+2} x^n \underbrace{- n c_n x^n - c_n x^n}_{-(n+1) c_n x^n} \right)$$

$$= 2c_2 - c_0 + (3(2)c_3 - 2c_1)x + (4(3)c_4 - 3c_2)x^2 + (5(4)c_5 - 4c_3)x^3$$

$$+ (6(5)c_6 - 5c_4)x^4 + (7(6)c_7 - 6c_5)x^5 + (8(7)c_8 - 7c_6)x^6$$

$$+ (9(8)c_9 - 8c_7)x^7$$

$c_2 = \frac{1}{2}c_0$

$12c_4 = 3c_2 \Rightarrow c_4 = \frac{3}{12}c_2 = \frac{1}{4}c_2 = \frac{1}{4}\cdot\frac{1}{2}c_0 = \frac{1}{8}c_0$

$30c_6 - 5c_4 = 0 \Rightarrow c_6 = \frac{1}{6}c_4 = \frac{1}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}c_0 = \frac{1}{48}c_0$

$56c_8 - 7c_6 = 0 \longrightarrow c_8 = \frac{1}{8}c_6 = \frac{1}{8}\cdot\frac{1}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}c_0 = \frac{1}{384}c_0$

$$\boxed{y_1 = c_0\left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \cdots\right.}$$

$$\begin{array}{r} 48 \\ \times\ 8 \\ \hline 384 \end{array}$$

$6c_3 - 2c_1 = 0 \longrightarrow 6c_3 = 2c_1 \longrightarrow c_3 = \frac{2}{6}c_1 = \frac{1}{3}c_1$

$20c_5 - 4c_3 = 0 \longrightarrow 20c_5 = 4c_3 \longrightarrow c_5 = \frac{1}{5}c_3 = \frac{1}{5}\cdot\frac{1}{3}c_1 = \frac{1}{15}c_1$

$c_7 = \frac{1}{7}\cdot\frac{1}{5}\cdot\frac{1}{3}c_1 = \frac{1}{105}c_1$

$c_9 =$

$$\boxed{y_2 = c_1\left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7 + \cdots\right)}$$

(4) Use an appropriate series about $x=0$ to find 2 solns of the ODE

$$\cos(x)\, y'' + y = 0$$

$$\left(\sum (-1)^n \frac{1}{2n} x^{2n}\right)\left(\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}\right) + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots\right)\left(2c_2 + 3\cdot 2 c_3 x + 4\cdot 3 c_4 x^2 + 5\cdot 4 c_5 x^3 + \cdots\right)$$

$$+ \sum_{n=0}^{\infty} c_n x^n$$

$$= 2c_2 + 6c_3 x + 6c_3 x^2 + 12 c_4 x^3 + 20 c_5 x^4 + 30 c_6 x^5 + 42 c_7 x^6$$

$$- \frac{1}{2}\cdot 2 c_2 x^2 - 3c_3 x^3 - 6 c_4 x^4 - 10 c_5 x^5$$

$$+ \frac{1}{12} c_2 x^4 + \frac{1}{4} c_3 x^5$$

$$\frac{14\cdot 12}{1 \cdot 12} + \frac{1}{12} = \frac{169}{12}$$

$$2c_2 + 6c_3 x + 5c_3 x^2 + 9c_3 x^3 + \frac{169}{12}$$

$$= + c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Don't trust my arithmetic

$$2 c_2 + c_0 + (6 c_3 + c_1) x + 12 c_4 x^2 + (20 c_5 - 2 c_3) x^3 + \left(30 c_6 - 5 c_4 + \frac{c_2}{12}\right) x^4 + \left(-9 c_5 + \frac{c_3}{4}\right) x^5 + \left(-14 c_6 + \frac{c_4}{2} - \frac{c_2}{360}\right) x^6 + \left(\frac{5 c_5}{6} - \frac{c_3}{120}\right) x^7 + \left(\frac{5 c_6}{4} - \frac{c_4}{60}\right) x^8 - \frac{c_5 x^9}{36} - \frac{c_6 x^{10}}{24} = 0$$

Note how this method breaks down because I truncated the series.

$$c_2 = -\frac{1}{2}c_0$$

$$c_4 = 0$$

$$30c_6 - 5c_4 + \frac{c_2}{12} = 0 \Longrightarrow 30 c_6 = -\frac{c_2}{12} \Longrightarrow c_6 = -\frac{c_2}{360} = -\frac{1}{360}\left(-\frac{1}{2}c_0\right) = \frac{1}{720}c_0$$

$$\boxed{y_1 = c_0\left[1 - \frac{1}{2}x^2 + \frac{1}{720}x^6 + \cdots\right]}$$

But if all they want is the first 3 terms, using degree-6 polynomials is adequate.

$$6c_3 + c_1 = 0 \Longrightarrow c_3 = -\frac{c_1}{6}$$

$$20c_5 = 2c_3 \Longrightarrow c_5 = \frac{1}{10}c_3 = \frac{1}{10}\cdot\left(-\frac{1}{6}\right)c_1 = -\frac{1}{60}c_1$$

$$\boxed{y_2 = c_1\left[1 - \frac{1}{6}x^3 - \frac{1}{60}x^5 + \cdots\right]}$$

(5) Solve the IVP with power series about the ordinary point $x=0$. Enter 1st 3 nonzero terms

$(x+3)y'' + 2y = 0$, s.t., $y(0)=0, y'(0)=1$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6$$

$$y' = \underbrace{c_1}_{=1 \text{ by IC's.}} + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + 6c_6 x^5$$

$$y'' = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4$$

$(x+3)y'' + 2y = 0$

Note: $y(0) = 0 = c_0$

$y'(0) = 1 = c_1$ So $y = x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots$

$(x+3)y'' + 2y$

$$= x(2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4) + 3(2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4)$$

$$+ 2(c_0 + c_1 x + c_2 x^2 + \dots + c_6 x^6) =$$

$$= 2c_2 x + 6c_3 x^2 + 12c_4 x^3 + 20c_5 x^4 + 30c_6 x^5 + 60c_2 + 18c_3 x + 36c_4 x^2 + 60c_5 x^3 + 90c_6 x^4$$

$$+ 2c_0 + 2c_1 x + 2c_2 x^2 + 2c_3 x^3 + 2c_4 x^4 + 2c_5 x^5 + 2c_6 x^6$$

$$= 60c_2 + 2c_0 + (\cancel{2c_2} + 18c_3 + 2c_1)x + (6c_3 + 36c_4 + \cancel{2c_2})x^2$$

$$+ (12c_4 + 60c_5 + 2c_3)x^3 + (20c_5 + 90c_6 + 2c_4)x^4 + \dots = 0$$

$c_0 = 0 \Rightarrow c_2 = 0$

$\longrightarrow 18c_3 + 2c_1 = 0 \longrightarrow 18c_3 = -2c_1 \Rightarrow c_3 = -\frac{1}{9}c_1$

$6c_3 + 36c_4 = 0 \longrightarrow 36c_4 = -6c_3 \longrightarrow c_4 = -\frac{1}{6}c_3 = -\frac{1}{6}\left(-\frac{1}{9}\right)c_1 = \frac{1}{54}c_1$

$\longrightarrow y(x) = c_1 x - \frac{1}{3}c_1 x + \frac{1}{54}c_1 x^3 + \dots$

By $y'(0) = 1 = c_1$, $\boxed{y = x - \frac{1}{3}x + \frac{1}{54}x^3 + \dots}$

(6) Is $x=0$ an ordinary point, regular singular pt or irregular singular point?

$$(e^x-1-x)y''+y=0 \longrightarrow$$

$$y''+\frac{1}{e^x-1-x}y=0$$

$P(x)=0$

$Q(x)=\dfrac{1}{e^x-1-x}$ $x=0$ is singular point.

Question is whether it's regular or irregular

$$\frac{x^2}{e^x-1-x}$$

$$=\frac{x^2}{1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}-1-x}=\frac{x^2}{\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots}$$

$$=\frac{x^2}{x^2\left(\frac{1}{2}+\frac{x}{3!}+\frac{x^2}{4!}+\dots\right)}=\frac{1}{\frac{1}{2}+\frac{x}{3!}+\frac{x^2}{4!}+\dots}$$

a power series for which may be computed, term by term, by long division, so that means it's analytic at $x = 0$.