Air containing 0.04% carbon dioxide is pumped into a room whose volume is 4,000 ft^3. The air is pumped in at a rate of 2,000 ft^3/min, and the circulated air is then pumped out at the same rate. If there is an initial concentration of 0.4% carbon dioxide, determine the subsequent amount $A(t)$, in ft^3, in the room at time t.

$V = 4000 ft$ in room

$(.04\% \ CO2)\left(\frac{2000 ft^3}{min}\right) = (.0004)(2000)\frac{ft^3}{min} = (.4)(2) = .8 \frac{ft^3}{min}$

Concentration of CO_2 in Room starts @ $.4\% \ CO_2$

$A(t)$ = amt of CO_2 in ft^3.

Start: $(0.4\%)(4000)\ ft^3 \ CO_2$

$(.004)(4000) = (4)(4) = \boxed{16 ft^3 \ CO_2 = A(0)}$

CO_2 OUT $= (\text{CONC OUT})\left(2000 \frac{ft^3}{min}\right) = \left(\frac{A(t)}{4000}\right)(2000) = \frac{1}{2}A(t)$

$\therefore \frac{dA}{dt} = .8 - \frac{1}{2}A(t) \longrightarrow \frac{dA}{dt} + \frac{1}{2}A(t) = .8 \longrightarrow$

$\mu = e^{\int \frac{1}{2} dt} = e^{\frac{1}{2}t}$

$(\mu A)' = .8\mu = .8e^{\frac{1}{2}t} \longrightarrow$

$\mu A = e^{\frac{1}{2}t}A(t) = 1.6e^{\frac{1}{2}t} + C \longrightarrow$

$A(t) = 1.6 + Ce^{-\frac{1}{2}t}$

$A(0) = 1.6 + C = 16 \longrightarrow$

$C = 14.4$

$\boxed{A(t) = 1.6 + 14.4e^{-\frac{1}{2}t}}$

$A(t) = ft^3$ of CO_2 in Room at time t, in minutes

What is the percent concentration of carbon dioxide at 10 minutes? (Round your answer to three decimal places.)

$A(10) = 1.6 + 14.4e^{-\frac{1}{2}(10)}$

$\text{conc} = \frac{A(10)}{4000}$ ≈ 0.0004242566092 $= .04242566092\%$

$\approx \boxed{.042\% = \text{CONCENTRATION AFTER 10 MIN}}$

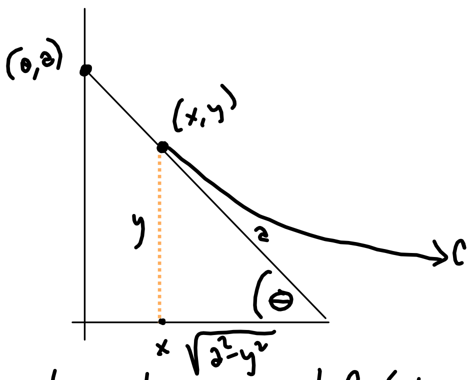
What is the steady-state, or equilibrium, percent concentration of carbon dioxide?

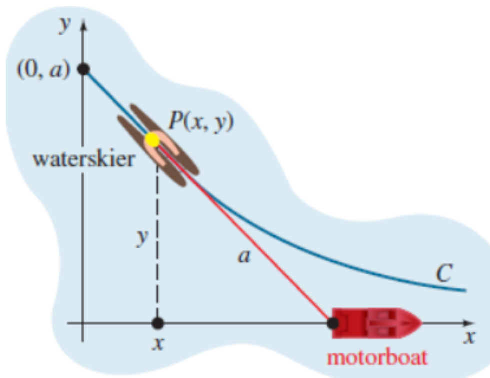
The steady-state concentration is the concentration of the incoming air, which is 0.04$

I just forgot the term used for the term that vanishes as t approaches infinity. Transitory?

Ephemeral? Fleeting? Whatever it's called, the decaying exponential term becomes vanishingly small for large values of *t*.

(2) A motorboat starts at the origin and moves to the right along the x-axis, pulling a waterskier along a curve called a **tractrix**. See the figure, below:





We derived this model (clumsily) in class.

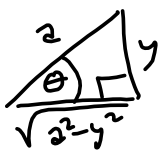
Evidently, $\tan\theta$ = − slope of the line from (x,y) to boat.

$\frac{dy}{dx} = -\tan\theta = -\frac{y}{\sqrt{a^2-y^2}}$ is our D.E., s.t. $y(0)=a$.

This eq'n is separable:

$$I = -\int \frac{\sqrt{a^2-y^2}}{y}\,dy = \int dx = \boxed{x + C = RHS}$$

$\sqrt{a^2-y^2}$ suggests the triangle



Then $\frac{\sqrt{a^2-y^2}}{y} = \cot\theta \Rightarrow$

$-\int \cot\theta\, dy$. Now, $y = a\sin\theta$ (pic) $\Rightarrow dy = a\cos\theta\, d\theta$

$$I = -\int \cot\theta\, a\cos\theta\, d\theta$$
$$= -a\int \frac{\cos^2\theta}{\sin\theta}\,d\theta = -a\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta = -a\int(\csc\theta - \sin\theta)\,d\theta$$

$$= -a\left[-\ln|\csc\theta + \cot\theta| - (-\cos\theta)\right]$$

$$= +a\ln|\csc\theta + \cot\theta| - a\cos\theta$$

$$= +a\ln\left|\frac{a}{y} + \frac{\sqrt{a^2-y^2}}{y}\right| - a\left(\frac{\sqrt{a^2-y^2}}{a}\right)$$

$$= +a\ln\left|\frac{a}{y} + \frac{\sqrt{a^2-y^2}}{y}\right| - \sqrt{a^2-y^2} = x + C$$

$y(0) = a$:

$$+a\ln\left|\frac{a}{a} + 0\right| + 0 = 0 + C$$

$$0 = C$$

$$\boxed{x = a\ln\left|\frac{a}{y} + \frac{\sqrt{a^2-y^2}}{y}\right| - \sqrt{a^2-y^2}}$$

Assume that the initial point on the y-axis is $(0, 6)$ and that the length of the rope is $x = 6$ ft.

$a = 6 \Longrightarrow$

$$\boxed{x = 6\ln\left|\frac{6}{y} + \frac{\sqrt{36-y^2}}{y}\right| - \sqrt{36-y^2}}$$

Book

$$\boxed{-6\ln\left|\frac{6}{y} - \frac{\sqrt{36-y^2}}{y}\right|} = -6\ln\left|\frac{6-\sqrt{36-y^2}}{y}\right|$$

$$= 6\ln\left|\frac{y}{6-\sqrt{36-y^2}}\right| = 6\ln\left|\frac{y(6+\sqrt{36-y^2})}{36-36+y^2}\right| = 6\ln\left|\frac{6y + y\sqrt{36-y^2}}{y^2}\right|$$

$$= 6\ln\left|\frac{6}{y} + \frac{\sqrt{36-y^2}}{y}\right| \checkmark$$

Chapter 3 Written Work #3

#3

$$\frac{dx}{dt} = k_1 x(\alpha - x)$$

$$\frac{dy}{dt} = k_2 xy$$

$$\frac{dx}{dt} = k_1(x(\alpha - x)) \Longrightarrow$$

$$\frac{dx}{x(\alpha - x)} = k_1 dt$$

$$\frac{1}{x(\alpha-x)} = \frac{A}{x} + \frac{B}{\alpha - x}$$

$$1 = A(\alpha - x) + Bx$$

$$x = \alpha \Longrightarrow 1 = \alpha B \Longrightarrow \boxed{B = \frac{1}{\alpha}}$$

$$x = 0 \Longrightarrow 1 = \alpha A \Longrightarrow A = \frac{1}{\alpha}$$

$$\frac{1}{\alpha}\int \frac{dx}{x} + \frac{1}{\alpha}\int \frac{dx}{\alpha - x} = k\int dt$$

$$\frac{1}{\alpha}\ln|x| + \frac{1}{\alpha}(-\ln(\alpha - x)) = \frac{1}{\alpha}\ln\left(\frac{x}{\alpha - x}\right) = k_1 t + C_1$$

$$\ln\left(\frac{x}{\alpha - x}\right) = \alpha k_1 t + \alpha C_1$$

$$\frac{x}{\alpha - x} = e^{\alpha k_1 t + \alpha C_1} = C_2 e^{\alpha k_1 t}$$

$$x = C_2(\alpha - x)e^{\alpha k_1 t} = c_2 \alpha e^{\alpha k_1 t} - c_2 x e^{\alpha k_1 t}$$

$$x + c_2 x e^{\alpha k_1 t} = c_2 \alpha e^{\alpha k_1 t}$$

$$\boxed{x = \frac{c_2 \alpha e^{\alpha k_1 t}}{1 + c_2 e^{\alpha k_1 t}}}$$

$$\frac{dy}{dt} = k_2 x y$$

$$\frac{dy}{y} = k_2 x \, dt = k_2 \left(\frac{c_2 \alpha e^{\alpha k_1 t}}{1 + c_2 e^{\alpha k_1 t}} \right) dt$$

$$u = 1 + c_2 e^{\alpha k_1 t}$$
$$du = c_2 \alpha k_1 e^{\alpha k_1 t} \, dt$$
$$dt = \frac{du}{c_2 \alpha k_1 e^{\alpha k_1 t}}$$

$$= \frac{k_2}{k_1} \ln\left|1 + c_2 e^{\alpha k_1 t}\right| + C_3$$

$$\ln|y| = \ln(y) = \ln\left(\left(1 + c_2 e^{\alpha k_1 t}\right)^{\frac{k_2}{k_1}}\right) + C_3$$

$$y = e^{C_3} \left(1 + c_2 e^{\alpha k_1 t}\right)^{\frac{k_2}{k_1}} = \boxed{c_4 \left(1 + c_2 e^{\alpha k_1 t}\right)^{\frac{k_2}{k_1}} = y}$$

You can re-label,
but don't need to.
Book sez $c_2 \left(1 + c_1 e^{\alpha k_1 t}\right)^{\frac{k_2}{k_1}}$ for y.
But it's very forgiving (correct) in the choices made for constants, except k_1 & k_2, which are specific.

#4

According to **Stefan's law of radiation** the absolute temperature T of a body cooling in a medium at constant absolute temperature T_m is given by

4. $$\frac{dT}{dt} = k(T^4 - T_m{}^4),$$

where k is a constant. Stefan's law can be used over a greater temperature range than Newton's law of cooling.

(a) Solve the differential equation.

Eq'n is Autonomous

$$\frac{dT}{T^4 - T_m^4} = k\,dt$$

$$T^4 - T_m^4 = (T^2 - T_m^2)(T^2 + T_m^2)$$

$$= (T - T_m)(T + T_m)(T^2 + T_m^2)$$

$$\frac{1}{T^4 - T_m^4} = \frac{A}{T - T_m} + \frac{B}{T + T_m} + \frac{CT + D}{T^2 + T_m^2}$$

$$1 = A(T + T_m)(T^2 + T_m) + B(T - T_m)(T^2 + T_m^2) + (CT + D)(T - T_m)(T + T_m)$$

$$1 = (AT + AT_m)(T^2 + T_m) + (BT - BT_m)(T^2 + T_m^2) + (CT + D)(T^2 - T_m^2)$$

$$= AT^3 + AT_mT + AT_mT^2 + AT_m^2 + BT^3 + BT_m^2T - BT_mT^2 - BT_m^3 + CT^3 - CT_m^2T + DT^2 - DT_m^2$$

$$1 = AT_m^2 - BT_m^3 - DT_m^2$$

$$0 = AT_mT + BT_m^3T - CT_m^2T$$

$$0 = AT_mT^2 - BT_mT^2 + DT^2$$

$$0 = AT^3 + BT^3 + CT^3$$

$$\left[\begin{array}{cccc|c} T_m^2 & -T_m^3 & 0 & -T_m^2 & 1 \\ T_m & T_m^3 & -T_m^2 & 0 & 0 \\ T_m & -T_m & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

Just use the technology. We'll get plenty of opportunities to do messy partial fractions in a 300- or 400-level math course, if we ever go that far.

I bet there are 2 or 3 errors messing it all up!

This is a great candidate for Maple:

$assume(x > y, y > 0)$

$A := \frac{1}{x^4 - y^4}$

$$A := \frac{1}{x\sim^4 - y\sim^4} \quad (2.3)$$

$convert(A, parfrac, x)$

$$-\frac{1}{2\,y\sim^2\,(x\sim^2 + y\sim^2)} + \frac{1}{4\,y\sim^3\,(x\sim - y\sim)} - \frac{1}{4\,y\sim^3\,(x\sim + y\sim)} \quad (2.4)$$

$$-\frac{1}{2\,Tm^2\,(T^2 + Tm^2)} + \frac{1}{4\,Tm^3\,(T - Tm)} - \frac{1}{4\,Tm^3\,(T + Tm)}$$

$$-\frac{1}{2\,Tm^2\,(T^2 + Tm^2)} + \frac{1}{4\,Tm^3\,(T - Tm)} - \frac{1}{4\,Tm^3\,(T + Tm)} \quad (2.5)$$

$normal(\%)$

$$\frac{1}{(T^2 + Tm^2)\,(T - Tm)\,(T + Tm)} \quad \checkmark \quad (2.6)$$

$$= \frac{1}{2T_m^2}\int\left(\frac{dT}{T^2+T_m^2}\right) + \frac{1}{4T_m^3}\int\left(\frac{dT}{T-T_m}\right) - \frac{1}{4T_m^3}\int\left(\frac{dT}{T+T_m}\right)$$

$$= \frac{1}{2T_m^4}\left[\frac{dT}{\frac{T^2}{T_m^2}+1}\right] + \frac{1}{4T_m^3}\ln|T-T_m| - \frac{1}{4T_m^3}\ln|T+T_m|$$

$$= \frac{1}{2T_m^3}\left[\frac{\frac{1}{T_m}dT}{\left(\frac{T}{T_m}\right)^2+1}\right] + \frac{1}{4T_m^3}\ln\left|\frac{T-T_m}{T+T_m}\right|$$

$$= \frac{1}{2T_m^3}\arctan\left(\frac{T}{T_m}\right) + \frac{1}{4T_m^3}\ln\left|\frac{T-T_m}{T+T_m}\right| = kt + \hat{C}$$

$$\Rightarrow \boxed{2\arctan\left(\frac{T}{T_m}\right) + \ln\left|\frac{T-T_m}{T+T_m}\right| = 4kT_m^3 + C} \quad (C = 4T_m^3\hat{C})$$