① $y = 3x^2 - 5x + 7 - 7e^{-3x} + 4xe^{-3x}$

$$\boxed{\begin{matrix} r=0,\ m=3 \\ r=-3,\ m=2 \end{matrix}}$$

② $y'' - 9y = 0$, s.t. $y(0) = 8$, $y'(0) = -6$

ⓐ $\boxed{y = c_1 e^{3x} + c_2 e^{-3x}}$

$y(0) = c_1 + c_2 = 8$

$y'(x) = 3c_1 e^{3x} - 3c_2 e^{-3x}$

$y'(0) = 3c_1 - 3c_2 = -6$

$$\left[\begin{array}{cc|c} 1 & 1 & 8 \\ 3 & -3 & -6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 8 \\ 1 & -1 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & -2 & -10 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 5 \end{array}\right]$$

$c_1 = 3$, $c_2 = 5$

$\boxed{y = 3e^{3x} + 5e^{-5x}}$

$-\frac{81}{20}x^5 + \frac{81}{10}x^6$

$+ \cdots]$

b) $y=\sum_{k=0}^{\infty} c_k x^k$, $y'=\sum_{k=1}^{\infty} k c_k x^{k-1} = \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k$,

$$y''=\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k \longrightarrow$$

$$y''-9y = \sum_{k=0}^{\infty}\left((k+2)(k+1)c_{k+2} - 9c_k\right)x^k = 0$$

$$\Longrightarrow c_{k+2} = \frac{9}{(k+2)(k+1)}c_k$$

$c_0 = 8,\ c_1 = -6$

$c_2 = \frac{9}{2}c_0 = \frac{9}{2}(8) = 36$

$c_3 = \frac{9}{(3)(2)}(-6) = \frac{-3^3 \cdot 2}{3 \cdot 2} = -9$

$c_4 = \frac{9}{4(3)}c_2 = \frac{9}{4 \cdot 3}\left(\frac{9}{2}\right)(8) = \frac{9^2}{4!}(8)$

$c_5 = \frac{9}{5(4)}c_3 = -\frac{81}{20}$

$c_6 = \frac{9^3}{6!}(8) = \frac{\cancel{9}^3\, 9^2}{\cancel{6} \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}}(\cancel{8}) = \frac{81}{10}$

$y_1 = \sum_{k=0}^{\infty} 8\left(\frac{9^k}{(2k)!}\right)x^{2k}$

$y_2 = \sum_{k=0}^{\infty}(-6)\frac{9^k}{(2k+1)!}x^{2k+1}$

$c_{2k} = (8)\left(\frac{9^{2k}}{(2k)!}\right)x^{2k}$

$c_{2k+1} = (-6)\frac{9^{2k+1}}{(2k+1)!}x^{2k+1}$

$$\boxed{y = 8 - 6x + 36x^2 - 9x^3 + 27x^4 + \cdots}$$

OR, $y = y_1 + y_2$

You can't always "see" the 2 independent power series sol'ns this easily, nor is their representation necessarily unique.

The series obtained by the odd terms is one solution. The series obtained by the even terms is the other.

Bonus

$$3\sum_{k=0}^{\infty}\frac{(3x)^k}{k!} + 5\sum_{k=0}^{\infty}\frac{(-3x)^k}{k!}$$

$$= 3\left(1+\frac{3^1}{1!}x + \frac{3^2}{2}x^2 + \frac{3^3}{3!}x^3 + \frac{3^4}{4!}x^4 + \frac{3^5}{5!}x^5 + \frac{3^6}{6!}x^6\right.$$

$$+5\left(1-\frac{3}{1}x+\frac{3^2}{2}x^2-\frac{3^3}{3!}x^3+\frac{3^4}{4!}x^4-\frac{3^5}{5!}x^5+\frac{3^6}{6!}x^6+\cdots\right]$$

$$= 3+5+(3-5)(3x)+(3+5)\frac{9}{2}x^2+(3-5)\frac{3^3}{3!}x^3+8\left(\frac{3^4}{4!}\right)x^4$$

$$-2\left(\frac{3^5}{5!}\right)x^5+8\left(\frac{3^6}{6!}\right)x^6+\cdots$$

$$= 8-6x+\frac{8(9)}{2}x^2-2\left(\frac{3^3}{6}\right)x^3+8\left(\frac{3^4}{4\cdot3\cdot2}\right)x^4-2\left(\frac{3^5}{5!}\right)x^5+8\left(\frac{3^6}{6!}\right)x^6$$

$$= 8-6x+36x^2-9x^3+27x^4-\cancel{2}\left(\frac{3^{\cancel{5}\,4}}{5\cdot4\cdot\cancel{3}\cdot\cancel{2}}\right)x^5+\cancel{8}\left(\frac{3^{\cancel{6}\,\cancel{5}\,4}}{\cancel{6}\,2\cdot5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}\right)x^6$$

$$= 8-6x+36x^2-9x^3+27x^4-\frac{81}{20}x^5+\frac{81}{10}x^6+\cdots$$

3a) $y''-9y=8e^t$ undetermined coefficients.

$y_c=c_1e^{3t}+c_2e^{-3t}$

$y_p=Ae^t=y_p''=y_p'$

$\rightarrow y_p''-9y_p=Ae^t-9Ae^t=-8Ae^t=8e^t \rightarrow$

$\boxed{A=\frac{8}{-8}=-1}$ $\rightarrow y=3e^{3t}+5e^{-3t}-e^t$

$y(0)=7,\ y'(0)=-7$

$y'=3c_1e^{3t}-3c_2e^{-3t}-e^t$

$y=y_c+y_p=c_1e^{3t}+c_2e^{-3t}-e^t$

$y(0)=7$ $\quad c_1+c_2-1=7$ $\qquad c_1+c_2=8$

$y'(0)=-7$ $\quad 3c_1-3c_2-1=-7$ $\qquad 3c_1-3c_2=-6$

$$\left[\begin{array}{cc|c}1&1&8\\3&-3&-6\end{array}\right]\sim\left[\begin{array}{cc|c}1&1&8\\0&-6&-30\end{array}\right]\sim\left[\begin{array}{cc|c}1&1&8\\0&1&5\end{array}\right]$$

$$\sim\left[\begin{array}{cc|c}1&0&3\\0&1&5\end{array}\right]$$

$\rightarrow \boxed{y=3e^{3x}+5e^{-3x}-e^x}$

3b $Ly = y'' - 9y = 8e^t$ $\qquad y(0)=7, y'(0)=-7$

$y_c = c_1 e^{3t} + c_2 e^{-3t}.$ $\qquad y_1 = e^{3t}, \; y_2 = e^{-3t}$

Variation of Parameters

$y_p = u_1 y_1 + u_2 y_2$, where $y_c = y_1 + y_2$

$y_1 u_1' + y_2 u_2' = 0$

$y_1' u_1' + y_2' u_2' = f(t)$

OPTIONAL

$\bar{u}' = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}, \; \bar{f} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$

$A\bar{u}' = \bar{f} \Longrightarrow$

$\bar{u}' = A^{-1}\bar{f}$

$= \frac{1}{|A|}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}\begin{bmatrix} 0 \\ f(t) \end{bmatrix}$

$= \frac{1}{|A|}\begin{bmatrix} -y_2 f(t) \\ y_1 f(t) \end{bmatrix}$

$= \frac{1}{|A|}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \frac{1}{w}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}$

$= \frac{1}{y_1 y_2' - y_2 y_1'}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}$

$A = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} e^{3t} & e^{-3t} \\ 3e^{3t} & -3e^{-3t} \end{bmatrix}$

Wronskian

$w = |A| = \begin{vmatrix} e^{3t} & e^{-3t} \\ 3e^{3t} & -3e^{-3t} \end{vmatrix} = -6$

$w_1 = \begin{vmatrix} 0 & y_2 \\ f(t) & y_2' \end{vmatrix} = -y_2 f(t) = -e^{-3x} \cdot 8e^x = -8e^{-2x}$

$w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(t) \end{vmatrix} = y_1 f(t) = e^{3x} \cdot 8e^x = 8e^{4x}$

$u_1' = \frac{w_1}{w} = \frac{-y_2 f(t)}{y_1 y_2' - y_2 y_1'} = \frac{1}{-6}\left(-8e^{-2x}\right)$

$= \boxed{\frac{4}{3}e^{-2x} = u_1'}$

$u_2' = \frac{w_2}{w} = -\frac{1}{6}\left(8e^{4x}\right) = \boxed{-\frac{4}{3}e^{4x} = u_2'}$

$\Longrightarrow y_p = y_1 u_1 + y_2 u_2 = e^{3x}\int_0^x$

$u_1 = \int \frac{w_1}{w}\,dx = \frac{4}{3}\int e^{-2x}\,dx = \frac{4}{3}\left(-\frac{1}{2}\right)e^{-2x} + C_1 = -\frac{2}{3}e^{-2x}$ Let $C=0$

$u_2 = \int \frac{w_2}{w}\,dx = -\frac{4}{3}\int e^{4x}\,dx = -\frac{4}{3}\cdot\frac{1}{4}e^{4x} + C_2 = -\frac{1}{3}e^{4x}$

$\Longrightarrow y_p = u_1 y_1 + u_2 y_2 = -\frac{2}{3}e^{-2x}e^{3x} - \frac{1}{3}e^{4x}e^{-3x} = \boxed{-e^x = y_p}$

3c GREEN'S $\qquad y''-9y=8e^{t}$, s.t. $y(0)=7,\ y'(0)=-7.$

$y_1=e^{3x},\ y_2=e^{-3x}$

$y_c=c_1e^{3x}+c_2e^{-3x}$ by previous work.

Now, we want y_c to solve $y''-9y=0$ st $y(0)=7, y'(0)=-7$

$$\left[\begin{array}{cc|c}1&1&7\\3&-3&-7\end{array}\right]\sim\left[\begin{array}{cc|c}1&1&7\\0&-6&-28\end{array}\right]\sim\left[\begin{array}{cc|c}1&0&\frac{7}{3}\\0&1&\frac{14}{3}\end{array}\right] \qquad -\frac{14}{3}+\frac{21}{3}=\frac{7}{3}$$

This says $\boxed{y_c=\frac{7}{3}e^{3x}+\frac{14}{3}e^{-3x}}$

$W=-6$, by previous work, using the fundamental solution $y_1=e^{3x}$, $y_2=e^{-3x}$. Then the particular solution y_p, satisfying $y_p(0)=y_p'(0)=0$ is given by

$$y_p=\int_0^x G(x,t)f(t)\,dt=\int_0^x G(x,t)\cdot 8e^{t}\,dt, \text{ where}$$

$$G(x,t)=\frac{y_1(t)y_2(x)-y_1(x)y_2(t)}{W(t)}=\frac{e^{3t}e^{-3x}-e^{3x}e^{-3t}}{-6}$$

$$=-\frac{1}{6}\left(e^{-3(x-t)}-e^{3(x-t)}\right)$$

$y''-9y=8e^{t}=f(t)$

$$y_p=\int_0^x G(x,t)f(t)\,dt=-\frac{1}{6}\int_0^x\left(e^{-3(x-t)}-e^{3(x-t)}\right)(8e^{t})\,dt$$

$$=-\frac{8}{6}\int_0^x\left(e^{-3(x-t)+t}-e^{3(x-t)+t}\right)dt$$

$$=-\frac{4}{3}\int_0^x\left(e^{-3x+3t+t}-e^{3x-3t+t}\right)dt$$

$$=-\frac{4}{3}\int_0^x\left(e^{-3x}e^{4t}-e^{3x}e^{-2t}\right)dt$$

$$=-\frac{4}{3}e^{-3x}\left[\frac{1}{4}e^{4t}\right]_0^x-\frac{4}{3}e^{3x}\left[\frac{1}{2}e^{-2t}\right]_0^x$$

$$=-\frac{1}{3}e^{-3x}\left[e^{4x}-e^{0}\right]+\frac{2}{3}e^{3x}\left[e^{-2x}-e^{0}\right]$$

$$=-\frac{1}{3}e^{x}+\frac{1}{3}e^{-3x}-\frac{2}{3}e^{x}+\frac{2}{3}e^{3x}$$

$$=-e^{x}+\frac{1}{3}e^{-3x}+\frac{2}{3}e^{3x}=y_p$$

Check:
$y_p(0)=-1+\frac{1}{3}+\frac{2}{3}=0$ ✓
$y_p'=-1-1+2=0$ ✓

$$y=y_c+y_p=\frac{7}{3}e^{3x}+\frac{14}{3}e^{-3x}+\frac{1}{3}e^{-3x}+\frac{2}{3}e^{3x}-e^{t}$$

$$=\boxed{y=3e^{3x}+5e^{-3x}-e^{t}}$$ ✓

3d) $y'' - 9y = 8e^{t}$, s.t. $y(0) = 7,\ y'(0) = -7$

$$s^2 F(s) - s y(0) - y'(0) - 9\big(F(s) = 8\left(\frac{1}{s-1}\right).\quad y(0)=7,\ y'(0) = -7$$

$$\Rightarrow s^2 F(s) - 7s + 7 - 9F(s) = \frac{8}{s-1}$$

$$(s^2-9)F(s) = \frac{8}{s-1} + 7s - 7$$

$$F(s) = \frac{8}{(s^2-9)(s-1)} + \frac{7s}{s^2-9} - \frac{7}{s^2-9}$$

See #5 work

$$= 8\left(-\frac{1}{8}\left(\frac{1}{s-1}\right) + \frac{1}{12}\left(\frac{1}{s-3}\right) + \frac{1}{24}\left(\frac{1}{s+3}\right)\right) + \frac{7s}{s^2-9} - \frac{7}{s^2-9}$$

$$= -\frac{1}{s-1} + \frac{2}{3}\left(\frac{1}{s-3}\right) + \frac{1}{3}\left(\frac{1}{s+3}\right) + \frac{7s}{s^2-9} - \frac{7}{s^2-9}$$

$$\Rightarrow y = -e^{t} + \frac{2}{3}e^{3t} + \frac{1}{3}e^{-3t} + 7\cosh(3t) - \frac{7}{3}\sinh(3t)$$

$$= -e^{t} + \frac{2}{3}e^{3t} + \frac{1}{3}e^{-3t} + \frac{7}{2}e^{3t} + \frac{7}{2}e^{-3t} - \frac{7}{3}\cdot\frac{1}{2}e^{3t} + \frac{7}{6}e^{-3t}$$

$$= -e^{t} + \left(\frac{2}{3} + \frac{7}{2} - \frac{7}{6}\right)e^{3t} + \left(\frac{1}{3} + \frac{7}{2} + \frac{7}{6}\right)e^{-3t}$$

$$= -e^{t} + \frac{4+21-7}{6}e^{3t} + \left(\frac{2+21+7}{6}\right)e^{-3t}$$

$$= \boxed{-e^{t} + 3e^{3t} + 5e^{-3t} = y}$$

)

3e

$$y'' - 9y = 8e^t \quad y(0) = 7,\ y'(0) = -7$$

$$y = \sum_0^\infty c_k x^k$$

$$y' = \sum_{k=1}^\infty k c_k x^{k-1}$$

$$y'' = \sum_{k=2}^\infty k(k-1) c_k x^{k-2}$$

$$y'' - 9y = \sum_{k=2}^\infty k(k-1) c_k x^{k-2} - 9\sum_{k=0}^\infty c_k x^k$$

$$= \sum_{k=0}^\infty (k+2)(k+1) c_{k+2} x^k - \sum_{k=0}^\infty 9 c_k x^k$$

$$= \sum_{k=0}^\infty \left((k+2)(k+1) c_{k+2} - 9c_k\right) x^k = 8\sum_{k=0}^\infty \frac{1}{k!} x^k$$

$$\sum_{k=0}^\infty \left((k+2)(k+1) c_{k+2} - 9c_k - \frac{8}{k!}\right) x^k = 0$$

$$\rightarrow (k+2)(k+1) c_{k+2} - 9c_k - \frac{8}{k!} = 0 \rightarrow$$

$$c_{k+2} = \frac{9c_k}{(k+2)(k+1)} + \frac{8}{(k+2)!}$$

$k=0 \quad c_0 = 7,$

$k=1 \quad c_1 = -7$

$k=2 \quad c_2 = \frac{9c_0}{(2)(1)} + \frac{8}{2!} = \frac{9(7)}{2} + \frac{8}{2} = \frac{63+8}{2} = \frac{71}{2}$

$k=3 \quad c_3 = \frac{9c_1}{3(2)} + \frac{8}{3!} = \frac{9(-7)}{6} + \frac{8}{6} = \frac{-63+8}{6} = \frac{-55}{6}$

$k=4 \quad c_4 = \frac{9c_2}{4\cdot 3} + \frac{8}{4!} = \frac{9c_2}{12} + \frac{8}{24} = \frac{9(\frac{71}{2})}{4\cdot 3} + \frac{8}{24} = \frac{639+8}{24} = \frac{647}{24}$

$$y = \boxed{7 - 7x + \frac{71}{2}x^2 - \frac{55}{6}x^3 + \frac{647}{24}x^4 + \ldots}$$

4) LRC - Circuit.

$$Li' + Ri + \frac{1}{C}\int_0^t i(\tau)\,d\tau = 10$$

$$L(sI(s) - i(0)) + RI(s) + \frac{1}{C}\int_0^t i(\tau)\cdot 1\,d\tau = \frac{10}{s}$$

$$LI(s)s + RI(s) + \frac{1}{C}\,I(s)\left(\frac{1}{s}\right) = \frac{10}{s}$$

$$LI(s)s + RI(s) + \frac{1}{C}\,I(s)\left(\frac{1}{s}\right) = \frac{10}{s}$$

$$I(s)\left(sL + R + \frac{1}{Cs}\right) = \frac{10}{s}$$

$$I(s)\left(\frac{1}{2}s + 10 + \frac{1}{\frac{1}{100}s}\right) = \frac{10}{s}$$

$$I(s)\left(\frac{s}{2} + 10 + \frac{100}{s}\right) = I(s)\left(\frac{s^2 + 20s + 200}{2s}\right) = \frac{10}{s}$$

$$I(s) = \frac{20s}{s(s^2+20s+200)} = \frac{20s}{s((s+10)^2 + 100)} = \frac{20}{(s+10)^2 + 10^2}$$

$$= 2\left(\frac{10}{(s+10)^2+10}\right) = \boxed{2e^{-10t}\sin(10t)}$$

5 $\frac{1}{(s-1)(s-3)(s+3)} = \frac{A}{s-1} + \frac{B}{s-3} + \frac{C}{s+3}$

$A(s+3)(s-3) + B(s-1)(s+3) + C(s-1)(s-3) = 1$

$s=1 \Rightarrow 4(-2)A = 1 \Rightarrow \boxed{A = -\frac{1}{8}}$

$s=3 \Rightarrow 2(6)B = 1 \Rightarrow \boxed{B = \frac{1}{12}}$

$s=-3 \Rightarrow -4(-6)C = 24C \Rightarrow 24C = 1 \Rightarrow \boxed{C = \frac{1}{24}}$

$-\frac{1}{8}\frac{1}{s-1} + \frac{1}{12}\left(\frac{1}{s-3}\right) + \frac{1}{24}\left(\frac{1}{s+3}\right)$

$\frac{8}{-8} = -1$

$\frac{8}{12} = \frac{2}{3}$

$\frac{8}{24} = \frac{1}{3}$