MAT 2560-R11 Midterm Test, Fall, 2024 Name________________________
120 points Covers Sections 1.1 – 4.4

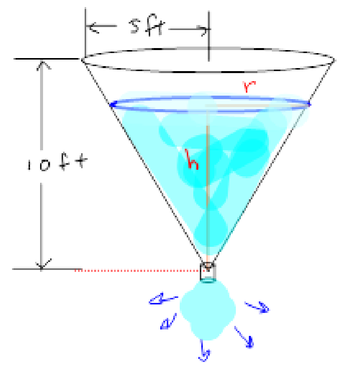
Please print your name in the space provided, above-right. Do all your work and circle all your final answers on the blank paper provided by the test proctors. Other than your name, I'm not looking at anything you write on these printed sheets. I'm only looking at the work you do on the blank sheets provided.

You don't need to write out the questions, but you do need to write out your answers as completely as possible.

There are 16 10-pointers on this test, plus a 10-point bonus question (because I don't think it got the greatest coverage). Work any 12 of them for up to 120 points, plus the 10-point bonus for up to 130 points.

Clearly indicate which problems not to grade by writing OMIT and putting a line through your work on that problem. Otherwise, I will just grade the first 120 points' worth I come to, and stop grading at #6.

1. A tank in the shape of an inverted cone of height 10 feet and width 5 feet starts out full of water. Then a petcock is opened and water begins draining through a 1-inch pipe at the bottom of the tank. Assuming there is no friction or other factors interfering with the water flow, the velocity of the outflow is the same as if the water drops had fallen from a height *h*, where *h* is the water level in the tank.



   a. (10 pts) Let V be the volume of water in the tank at time t. By high-school formula, $V = \frac{1}{3}\pi r^2 h$, where V, r, and h are all considered functions of t. Use the dimensions of the cone to make a substitution for r in terms of h. Then differentiate both sides of the resulting equation implicitly with respect to t. This gives you an expression for $\frac{dV}{dt}$.

   b. (10 pts) The way to think of potential energy P_E is by how much work it would take to lift the water from the bottom of the tank to the top of the water at height h. mg is the force (mass times acceleration due to gravity) and h is the distance over which that force acts, so $P_E = mgh$, where $g = 32\frac{\text{ft}}{\text{s}^2}$. For the same mass of water, the kinetic energy P_K is given by $P_K = \frac{1}{2}mv^2$, where v is the velocity of the water out the pipe. Solve $P_E = P_K$ for v. Assume v is positive. Save this result. You'll need it in part c.

   c. (10 pts) There is another expression for $\frac{dV}{dt}$, since it's the product of the velocity v of the water leaving the tank and the cross-sectional area of the pipe. Write this product. Remember to convert the radius of the pipe to feet. Remember that a 2-inch pipe has a 2-inch *diameter*. Use the value for v that you obtained in part b.

   d. (10 pts) Equate the two expressions for $\frac{dV}{dt}$. This gives a separable differential equation in $h = h(t)$. Solve the initial value problem thus constructed. Use the initial conditions suggested by the tank starting out full.

2. (10 pts) Use substitution to solve the differential equation $(y - 3x - 1)y' = 1$.

3. Consider the autonomous differential equation $\frac{dP}{dt} = P(100 - P)$
   a. (10 pts) Build a phase portrait for the differential equation.

   b. (10 pts) From your phase portrait, give a rough sketch of what the direction field or slope field of the equation looks like.

   c. (10 pts) Given $P(0) = 50$, find the unique solution P of the differential equation.

   d. (10 pts) Sketch your solution by superimposing it on the direction field in part b. It's the solution passing through (0, 50). Hint: The inflection point is at (t, P), where P is exactly half of the "carrying capacity."

4. (10 pts) Given that $y_1 = x^2$ is a solution of the equation $x^2 y'' + 2xy' - 6y = 0$, use reduction-of-order to find a second solution of the form $y_2 = uy_1$. You may check your work using the formula $y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$, but I want to see you reason your way to the solution.

5. (10 pts) Solve the separable ODE $\frac{dy}{dx} = \frac{\sin(3x)}{\sqrt{y}}$

Bonus (10 pts) Sometimes, you can't find a closed-form antiderivative. But you can still write the solution in terms of the unevaluated integral, using the initial condition. Solve the separable ODE $\frac{dy}{dx} = \frac{\sin(x^2)}{\sqrt{y}}$ subject to $y(0) = 4$.

6. (10 pts) Show that the differential equation $(2x + 2y)dx + (2x - 3y^2)dy = 0$ is exact and then solve it.

7. (10 pts) Solve the homogeneous linear 2nd-order ODE $y'' - 5y' + 6y = 0$.

8. (10 pts) Solve the nonhomogeneous linear 2nd-order ODE $y'' - 5y' + 6y = 2x + 7$. You already have the complementary solution (#5 answer), so all you need to do, here is find a particular solution.

9. (10 pts) Solve the homogeneous linear 3rd-order ODE $y''' - 6y'' + 21y' - 26y = 0$. Hint: $r = 2$ is a zero of $r^3 - 6r^2 + 21r - 26$.

10. (10 pts) Solve the homogeneous linear ODE $y'' - 10y' + 25y = 0$