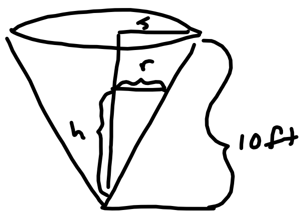
MIDTERM SOLUTIONS

(1)



$\frac{r}{h} = \frac{5}{10} \Longrightarrow$

$r = \frac{h}{2} \Longrightarrow$

(a) $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3 \Longrightarrow$

$$\boxed{\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}}$$

(b) $\frac{1}{2}mv^2 = mgh$ Little v is velocity Big V is Volume.

$\Longrightarrow v^2 = 2gh$

$\Longrightarrow v = \pm\sqrt{2gh}$. $v > 0$ & $g = 32 \Longrightarrow$

$v = \sqrt{2(32)h} = 8\sqrt{h} = \boxed{8h^{\frac{1}{2}} = \text{velocity}}$

(c) $\frac{dV}{dt} = (\text{velocity})(\text{cross-section})$

$\frac{8}{144} = \frac{4}{72} = \frac{2}{36} = \frac{1}{18}$

$= \left(8h^{\frac{1}{2}}\right)\left(\pi\left(\frac{1}{24}ft\right)^2\right) = \frac{8\pi h^{\frac{1}{2}}}{576} = \frac{\pi h^{\frac{1}{2}}}{72}$

Diameter = 1''

$\Longrightarrow$ Radius = $\frac{1}{2}'' \cdot \frac{1ft}{12''}$

$= \frac{1}{24}ft$

(d) $\frac{dV}{dt} = \frac{dV}{dt}$

$\frac{1}{4}\pi h^2 \frac{dh}{dt} = \frac{\pi}{72} h^{\frac{1}{2}} \Longrightarrow$

$\frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{\pi}{72} h^{\frac{1}{2}} = \frac{1}{18} h^{-\frac{3}{2}} \Longrightarrow$

$\cancel{\pi}$ $\cancel{4}$ $\cancel{8}$ $\cancel{576}$ $\cancel{288}$ $\cancel{144}$ 72

$h^{\frac{3}{2}} dh = \frac{1}{18} dt \Longrightarrow$

$\frac{2}{5} h^{\frac{5}{2}} = \frac{1}{18} t + C$

$h(0) = 10 \Longrightarrow$

$10^{\frac{5}{2}} = 10^{2+\frac{1}{2}} = 10^2 10^{1/2} = 100\sqrt{10}$

$\frac{2}{5}(10)^{\frac{5}{2}} = C$

$\left(\frac{2}{5}\right)(100\sqrt{10}) = 40\sqrt{10} = C \Longrightarrow$

$\frac{2}{5} h^{\frac{5}{2}} = \frac{1}{18} t + 40\sqrt{100}$

$h^{\frac{5}{2}} = \frac{5}{36} t + 20(5)\sqrt{10}$

$$\boxed{h = \left(\frac{5}{36} t + 100\sqrt{10}\right)^{\frac{2}{5}}}$$

② $(y-3x-1)y' = 1$

Let $u = y-3x-1$. Then

$\frac{du}{dx} = \frac{dy}{dx} - 3 \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + 3 \Rightarrow$

$u\left(\frac{du}{dx}+3\right) = u\frac{du}{dx} + 3u = 1 \Rightarrow$

$u\left(\frac{du}{dx}\right) = 1-3u \Rightarrow$

$\frac{u\,du}{1-3u} = dx$. Let $v = 1-3u \Rightarrow dv = -3du \Rightarrow$ $du = \frac{dv}{-3}$

$\Rightarrow v-1 = -3u \Rightarrow u = \frac{1-v}{3} \Rightarrow$

$\int \frac{\left(\frac{1-v}{3}\right)\left(\frac{dv}{-3}\right)}{v} = -\frac{1}{9}\int \frac{(1-v)dv}{v} = -\frac{1}{9}\int\left(\frac{1}{v} - 1\right) dv =$

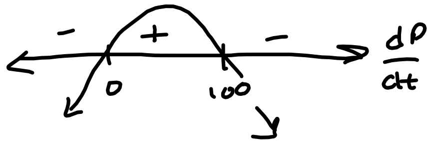
$= -\frac{1}{9}\left[\ln|v| - v\right] + C = -\frac{1}{9}\ln(1-3u) - \frac{(1-3u)}{9} + C$

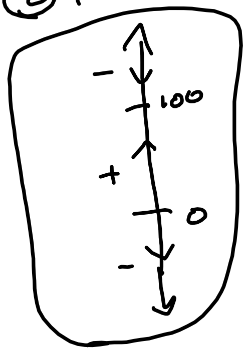
$= \frac{1}{9}\ln\left(1-3(y-3x-1)\right) - \frac{1}{9}\left(1-3(y-3x-1)\right) + C$

$\boxed{= \frac{1}{9}\ln\left(-3y+9x+4\right) - \frac{1}{9}\left(-3y+9x+4\right) + C}$

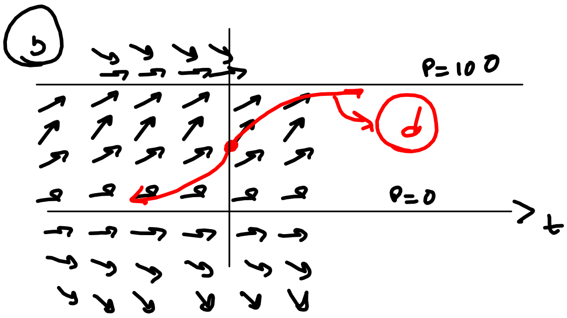
(3) $\dfrac{dP}{dt} = P(100-P)$

(a) Phase Portrait





(b)



(c) $\dfrac{dP}{dt} = P(100-P) \Longrightarrow$

$$\frac{dP}{P(100-P)} = dt$$

$$\frac{1}{P(100-P)} = \frac{A}{P} + \frac{B}{100-P}$$

$$A(100-P) + BP = 1$$

$P=0$: $\quad 100A = 1$, $\quad A = \frac{1}{100}$

$P=100$: $\quad 100B = 1$, $\quad B = \frac{1}{100}$

$$\therefore \int \frac{dP}{P(100-P)} = \frac{1}{100}\left[\ln(P) - \ln(100-P)\right] = t + C$$

$P(0) = 50 \Longrightarrow$

$$\frac{1}{100}\left[\ln(50) - \ln(50)\right] = C \Longrightarrow C = 0\,!$$

$$\Longrightarrow \frac{1}{100}\left[\ln(P) - \ln(100-P)\right] = t \Longrightarrow$$

$$\ln\left(\frac{P}{100-P}\right) = 100t \Longrightarrow$$

$$\frac{P}{100-P} = e^{100t} \Longrightarrow$$

$$P = (100-P)e^{100t} = 100e^{100t} - Pe^{100t} \Longrightarrow$$

$$P + Pe^{100t} = 100e^{100t}$$

$$\Longrightarrow P(1+e^{100t}) = 100e^{100t}$$

$$\Longrightarrow P = \frac{100e^{100t}}{1+e^{100t}} \quad \text{OR} \quad \boxed{P(t) = \frac{100}{e^{-100t}+1}}$$

4) $y_1 = x^2$ is given as a sol'n of

$x^2y'' + 2xy' - 6y = 0$. Use red-of-order to find the 2nd sol'n.

We look for $u(x) \ni u(x)y_1(x) = y_2(x)$ solves the eq'n.

Re-write eq'n:

$$y'' + \frac{2}{x}y' - \frac{6}{x^2}y = 0$$

Want $u(x) \ni y_2 = u(x)y_1(x) = ux^2$ is sol'n.

$y_2' = u'x^2 + 2ux$

$y_2'' = u''x^2 + 2u'x + 2u'x + +2u$

$\to y_2'' + \frac{2}{x}y_2' - \frac{6}{x^2}y_2$

$= u''x^2 + 2u'x + 2u'x + 2u \quad + \frac{2}{x}(u'x^2 + 2ux) - \frac{6}{x^2}ux^2$

$= u''x^2 + 4u'x + \frac{2}{x}u'x^2 + 2u + \frac{2}{x}2ux - \frac{6}{x^2}ux^2$

$= u''x^2 + 4u'x + \frac{2}{x}u'x^2 + 2u + 4u - 6u$

$= x^2u'' + (4x + 2x)u' = 0$

$= x^2u'' + 6xu' = 0 \to$

$x^2u'' = -6xu' \to$

$\frac{u''}{u'} = -\frac{6}{x} \to$

$\ln|u'| = -6\ln|x| + C \to$

$\ln|u'| = \ln|x^{-6}| + C \to$

$u' = Kx^{-6} = x^{-6} \to$

$u = -\frac{1}{5}x^{-5} + C = -\frac{1}{5}x^{-5} \to$

$y_2 = ux^2 = -\frac{1}{5}x^{-5}\cdot x^2 = -\frac{1}{5}x^{-3}$

$\Rightarrow$ Sol'n is $\boxed{c_1x^{-3} + c_2x^2 = y}$

5 $\frac{dy}{dx} = \frac{\sin(3x)}{\sqrt{y}} \longrightarrow$

$$\sqrt{y}\,dy = \sin(3x)\,dx$$

$$\frac{2}{3}y^{\frac{3}{2}} = -\frac{1}{3}\cos(3x) + c$$

$$y = \left(\frac{3}{2}\left(-\frac{1}{3}\right)\cos(3x) + \frac{3}{2}c\right)^{\frac{2}{3}}$$

$$\boxed{y = \left(-\frac{1}{2}\cos(3x) + \hat{C}\right)^{\frac{2}{3}}}$$

BONUS Solve $\frac{dy}{dx} = \frac{\sin(x^2)}{\sqrt{y}} \longrightarrow$

$$\sqrt{y}\,dy = \sin(x^2)\,dx$$

$$\frac{2}{3}y(x)^{\frac{3}{2}} - \frac{2}{3}y(0)^{\frac{3}{2}} = \int_0^x \sin(t^2)\,dt$$

$$\Longrightarrow y(x)^{\frac{3}{2}} - 4^{\frac{3}{2}} = \frac{3}{2}\int_0^x \sin(t^2)\,dt$$

$$y(x)^{\frac{3}{2}} = \frac{3}{2}\int_0^x \sin(t^2)\,dt + 8$$

$$\boxed{y(x) = \left(\frac{3}{2}\int_0^x \sin(t^2)\,dt + 8\right)^{\frac{2}{3}}}$$

6 $(2x+2y)\,dx + (2x-3y^2)\,dy = 0$

is exact:

$M_y = 2 = N_x = 2$ ✓

$f(x,y) = \int (2x+2y)\,dx + g(y)$

$f_y = N = \frac{\partial}{\partial y}\int (2x+2y)\,dx + g'(y)$

$= \frac{\partial}{\partial y}\left[x^2+2yx\right] + g'(y)$

$= 2x + g'(y) = N = 2x - 3y^2 \Rightarrow$

$g'(y) = -3y^2 \Rightarrow$

$g(y) = -y^3 \Rightarrow$

$f(x,y) = \boxed{x^2+2xy-y^3 = C \text{ is implicit sol'n}}$

(7) $y'' - 5y' + 6y = 0$

$(r-3)(r-2) = 0 \longrightarrow$

$$\boxed{y = c_1 e^{3x} + c_2 e^{2x}}$$

(8) $y'' - 5y' + 6y = 2x + 7$

$$\boxed{y_h = c_1 e^{3x} + c_2 e^{2x}}$$

$y_p: = Ax + B$

$y_p' = A$

$y_p'' = 0$

$y_p'' - 5y_p' + 6y = -5A + 6(Ax + B) = 2x + 7$

$\Longrightarrow -5A + 6B = 7$

$6A = 2$

$A = \frac{2}{6} = \boxed{\frac{1}{3} = A}$

$-5(\frac{1}{3}) + 6B = 7$

$-\frac{5}{3} + 6B = 7$

$-5 + 18B = 21$

$18B = 26$

$$\boxed{B = \frac{13}{9}}$$

$\therefore \boxed{y_p = \frac{1}{3}x + \frac{13}{9}}$

& $\boxed{y = y_h + y_p}$ is the soln.

(9) We solve $y''' - 6y'' + 21y' - 26y = 0$, given $\boxed{r=2}$ is a root of the auxiliary eqn:

$r^3 - 6r^2 + 21r - 26$

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 21 & -26 \\ & & 2 & -8 & 26 \\ \hline & 1 & -4 & 13 & 0 \end{array}$$

$x^2 - 4x + 13 = x^2 - 4x + 2^2 - 4 + 13 = (x-2)^2 + 9 = 0$

$\Rightarrow \boxed{r = 2 \pm 3i}$

$$\boxed{y = c_1 e^{2x} + e^{2x}\left(c_2 \cos(3x) + c_3 \sin(3x)\right)}$$

(10) $y'' - 10y' + 25y = 0$ is solved:

$r^2 - 10r + 25 = 0 \Rightarrow$

$(r-5)^2 = 0 \Rightarrow$

$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$ is sol'n