Chapter 3 Written Work #3

#3

$$\frac{dx}{dt} = K_1 x(\alpha - x)$$

$$\frac{dy}{dt} = K_2 x y$$

$$\frac{dx}{dt} = K_1(x(\alpha - x)) \Longrightarrow$$

$$\frac{dx}{x(\alpha - x)} = K_1 dt$$

$$\frac{1}{x(\alpha - x)} = \frac{A}{x} + \frac{B}{\alpha - x}$$

$$1 = A(\alpha - x) + Bx$$

$$x = \alpha \Longrightarrow 1 = \alpha B \Longrightarrow \boxed{B = \frac{1}{\alpha}}$$

$$x = 0 \Longrightarrow 1 = \alpha A \Longrightarrow A = \frac{1}{\alpha}$$

$$\frac{1}{\alpha}\int \frac{dx}{x} + \frac{1}{\alpha}\int \frac{dx}{\alpha - x} = K\int dt$$

$$\frac{1}{\alpha}\ln|x| + \frac{1}{\alpha}(-\ln(\alpha - x)) = \frac{1}{2}\ln\left(\frac{x}{\alpha - x}\right) = K_1 t + C_1$$

$$\ln\left(\frac{x}{\alpha - x}\right) = \alpha K_1 t + \alpha C_1$$

$$\frac{x}{\alpha - x} = e^{\alpha K_1 t + \alpha C_1} = C_2 e^{\alpha K_1 t}$$

$$x = \alpha C_2 e^{\alpha K_1 t} - x C_2 e^{\alpha K_1 t}$$

$$x(1 + C_2 e^{\alpha K_1 t}) = \alpha C_2 e^{\alpha K_1 t}$$

$$x = \frac{\alpha C_2 e^{\alpha K_1 t}}{1 + C_2 e^{\alpha K_1 t}}$$

$$\longrightarrow \frac{dy}{dt} = K_2 x y = K_2 \left(\frac{\alpha C_2 e^{\alpha K_1 t}}{1 + C_2 e^{\alpha K_1 t}}\right) y \longrightarrow$$

$$\frac{dy}{y} = K_2 \frac{\alpha C_2 e^{\alpha K_1 t}}{1 + C_2 e^{\alpha K_1 t}} dt \longrightarrow$$

$$\ln|y| = K_2 \int \frac{\cancel{\alpha C_2 e^{\alpha K_1 t}}}{u} \cdot \frac{du}{\cancel{C_2 \alpha} K_1 \cancel{e^{\alpha K_1 t}}} = K_2 \int \frac{1}{K_1} \frac{du}{u}$$

$$u = 1 + C_2 e^{\alpha K_1 t}$$

$$du = C_2 \alpha K_1 e^{\alpha K_1 t} dt$$

$$dt = \frac{du}{C_2 \alpha K_1 e^{\alpha K_1 t}}$$

$$= \frac{K_2}{K_1} \ln|1 + C_2 e^{\alpha K_1 t}| + C_3$$

$$\ln|y| = \ln(y) = \ln\left(\left(1 + C_2 e^{\alpha K_1 t}\right)^{\frac{K_2}{K_1}}\right) + C_3 \longrightarrow$$

$$y = e^{C_3} (1 + C_2 e^{\alpha K_1 t})^{\frac{K_2}{K_1}} = \boxed{C_4 (1 + C_2 e^{\alpha K_1 t})^{\frac{K_2}{K_1}} = y}$$

You can re-label,
but don't need to.
Book see $C_2 (1 + C_1 e^{\alpha K_1 t})^{\frac{K_2}{K_1}}$ for y.