#9

I recommend running this both with and without the following *assume* command.

$assume(s > 0)$

$with(LinearAlgebra):$

$with(inttrans):$

We take the Laplace Transform of the system:

$x'' + y'' = e^{2t}$

$2x' + y'' = e^{-2t}$

subject to $x(0) = x'(0) = y(0) = y'(0) = 0$.

$eqn1 := diff(diff(x(t), t), t) + diff(diff(y(t), t), t) = \exp(2 \cdot t)$

$$eqn1 := \frac{d^2}{dt^2}\, x(t) + \frac{d^2}{dt^2}\, y(t) = e^{2t} \tag{1.1}$$

$eqn2 := 2 \cdot diff(x(t), t) + diff(diff(y(t), t), t) = \exp(-2 \cdot t)$

$$eqn2 := 2\, \frac{d}{dt}\, x(t) + \frac{d^2}{dt^2}\, y(t) = e^{-2t} \tag{1.2}$$

$laplace(eqn1, t, s)$

$$s^2\, \mathcal{L}(x(t), t, s) - D(x)(0) - s\, x(0) + s^2\, \mathcal{L}(y(t), t, s) - D(y)(0) - s\, y(0) = \frac{1}{s-2} \tag{1.3}$$

$subs([x(0) = 0, D(x)(0) = 0, y(0) = 0, D(y)(0) = 0], \%)$

$$s^2\, \mathcal{L}(x(t), t, s) + s^2\, \mathcal{L}(y(t), t, s) = \frac{1}{s-2} \tag{1.4}$$

$laplace(eqn2, t, s)$

$$2\, s\, \mathcal{L}(x(t), t, s) - 2\, x(0) + s^2\, \mathcal{L}(y(t), t, s) - D(y)(0) - s\, y(0) = \frac{1}{s+2} \tag{1.5}$$

$subs([x(0) = 0, D(x)(0) = 0, y(0) = 0, D(y)(0) = 0], \%)$

$$2\, s\, \mathcal{L}(x(t), t, s) + s^2\, \mathcal{L}(y(t), t, s) = \frac{1}{s+2} \tag{1.6}$$

This gives us 2 equations in the unknowns *L*(*x*) and *L*(*y*). The coefficient matrix for the system is:

$A := \left\langle s^2, s^2, \frac{1}{s-2}; 2 \cdot s, s^2, -\frac{1}{s-2} \right\rangle$

$$A := \begin{bmatrix} s^2 & s^2 & \frac{1}{s-2} \\ 2\, s & s^2 & -\frac{1}{s-2} \end{bmatrix} \tag{1.7}$$

The Reduced-Row-Echelon Form of the coefficient matrix is:

Row_Reduced_A := *ReducedRowEchelonForm*(*A*)

$$Row\_Reduced\_A := \begin{bmatrix} 1 & 0 & \frac{2}{s\,(s-2)^2} \\ 0 & 1 & -\frac{s+2}{s^2\,(s-2)^2} \end{bmatrix} \quad \textbf{(1.8)}$$

This allows us to read the transforms of *x* and *y* directly.

The 1st row, 3rd column of the matrix is *L*(*x*). The 2nd row, 3rd column of the matrix is *L*(*y*):

Compute the inverse Laplace Transforms of each.

This gives *x*(*t*):

invlaplace(*Row_Reduced_A*[1, 3], *s*, *t*)

$$\frac{1}{2} + \frac{e^{2t}\,(-1+2t)}{2} \quad \textbf{(1.9)}$$

expand(%)

$$\frac{1}{2} - \frac{(e^{t})^2}{2} + (e^{t})^2\,t \quad \textbf{(1.10)}$$

This gives *y*(*t*):

invlaplace(*Row_Reduced_A*[2, 3], *s*, *t*)

$$-\frac{3}{4} - \frac{t}{2} - \frac{e^{2t}\,(4t-3)}{4} \quad \textbf{(1.11)}$$

expand(%)

$$-\frac{3}{4} - \frac{t}{2} - (e^{t})^2\,t + \frac{3\,(e^{t})^2}{4} \quad \textbf{(1.12)}$$

In other words, $x(t) = \frac{1}{2} + t\,e^{2t} - \frac{e^{2t}}{2}$, $y(t) = \frac{3}{4} - \frac{t}{2} - e^{2t}\,t + \frac{3\,e^{2t}}{4}$.

Try to Teach Some Linear Algebra Moves, One Last Time

We use Gauss-Jordan to obtain the Reduced Row-Echelon Form of the matrix *A* so that you may read the solution directly from the matrix:

A

$$\begin{bmatrix} s^2 & s^2 & \frac{1}{s-2} \\ 2\,s & s^2 & -\frac{1}{s-2} \end{bmatrix} \tag{2.1}$$

Closer to how you'd code it, referring to entries by their location in the matrix...

This isn't strictly necessary, but as a matter of taste, you may want the lower power of *s* to start out in the top left corner. But it really makes no difference.

Using the indices of the matrix entries suggests (I hope) a nested loop that can perform the same tricks for arbitrary *n* x *n* matrices.

$RowOperation(A, [1, 2])$

$$\begin{bmatrix} 2\,s & s^2 & -\frac{1}{s-2} \\ s^2 & s^2 & \frac{1}{s-2} \end{bmatrix} \tag{2.1.1}$$

Use division to put a '1' in the top left corner. (Divide Row 1 by *A*[1,1].)

$RowOperation\left(\%, 1, \frac{1}{\%[1, 1]}\right)$

$$\begin{bmatrix} 1 & \frac{s}{2} & -\frac{1}{2\,s\,(s-2)} \\ s^2 & s^2 & \frac{1}{s-2} \end{bmatrix} \tag{2.1.2}$$

Eliminate the entry in the 2nd row, beneath the leading '1.' Basically, Replace *Row* 2 by $-s^2 Row\,1 + Row\,2$.

$RowOperation(\%, [2, 1], -\%[2, 1])$

$$\begin{bmatrix} 1 & \frac{s}{2} & -\frac{1}{2\,s\,(s-2)} \\ 0 & s^2 - \frac{1}{2}\,s^3 & \frac{1}{s-2} + \frac{s}{2\,(s-2)} \end{bmatrix} \tag{2.1.3}$$

Make the leading entry in *Row* 2 into a '1' by dividing *Row* 2 by its leading (nonzero) coefficient.

$RowOperation\left(\%, 2, \frac{1}{\%[2, 2]}\right)$

$$\begin{bmatrix} 1 & \frac{s}{2} & -\frac{1}{2\,s\,(s-2)} \\ 0 & 1 & \frac{\frac{1}{s-2}+\frac{s}{2\,(s-2)}}{s^2-\frac{1}{2}\,s^3} \end{bmatrix} \quad \textbf{(2.1.4)}$$

simplify(%)

$$\begin{bmatrix} 1 & \frac{s}{2} & -\frac{1}{2\,s\,(s-2)} \\ 0 & 1 & \frac{-s-2}{s^2\,(s-2)^2} \end{bmatrix} \quad \textbf{(2.1.5)}$$

Eliminate the entry above the leading entry of *Row* 2:

RowOperation(%, [1, 2], −%[1, 2])

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2\,s\,(s-2)} - \frac{s\left(\frac{1}{s-2}+\frac{s}{2\,(s-2)}\right)}{2\left(s^2-\frac{1}{2}\,s^3\right)} \\ 0 & 1 & \frac{\frac{1}{s-2}+\frac{s}{2\,(s-2)}}{s^2-\frac{1}{2}\,s^3} \end{bmatrix} \quad \textbf{(2.1.6)}$$

simplify(%)

$$\begin{bmatrix} 1 & 0 & \frac{2}{s\,(s-2)^2} \\ 0 & 1 & \frac{-s-2}{s^2\,(s-2)^2} \end{bmatrix} \quad \textbf{(2.1.7)}$$

Closer to how you'd do it by hand...

$RowOperation\left(A, 1, \frac{1}{s^2}\right)$

$$\begin{bmatrix} 1 & 1 & \frac{1}{s^2\,(s-2)} \\ 2\,s & s^2 & -\frac{1}{s-2} \end{bmatrix} \quad \textbf{(2.2.1)}$$

RowOperation(%, [2, 1], $-2 \cdot s$)

$$\begin{bmatrix} 1 & 1 & \frac{1}{s^2\,(s-2)} \\ 0 & s^2-2\,s & -\frac{1}{s-2}-\frac{2}{s\,(s-2)} \end{bmatrix} \tag{2.2.2}$$

$RowOperation\left(\%, 2, \frac{1}{s^2-2\cdot s}\right)$

$$\begin{bmatrix} 1 & 1 & \frac{1}{s^2\,(s-2)} \\ 0 & 1 & \frac{-\frac{1}{s-2}-\frac{2}{s\,(s-2)}}{s^2-2\,s} \end{bmatrix} \tag{2.2.3}$$

$RowOperation(\%, [1, 2], -1)$

$$\begin{bmatrix} 1 & 0 & \frac{1}{s^2\,(s-2)}-\frac{-\frac{1}{s-2}-\frac{2}{s\,(s-2)}}{s^2-2\,s} \\ 0 & 1 & \frac{-\frac{1}{s-2}-\frac{2}{s\,(s-2)}}{s^2-2\,s} \end{bmatrix} \tag{2.2.4}$$

Now just clean up the output and read the Laplace Transform for *x* and *y* off the 3rd column:
simplify(%)

$$\begin{bmatrix} 1 & 0 & \frac{2}{s\,(s-2)^2} \\ 0 & 1 & \frac{-s-2}{s^2\,(s-2)^2} \end{bmatrix} \tag{2.2.5}$$