# Final Review for Final

$Ly = y'' - 4y' + 3y = te^t$

$r^2 - 4r + 3 = (r-3)(r-1)$ $y_1 = e^{3t}$ $y_2 = e^t$ $f(t) = te^t$

Variation of Parameters

$y_p = u_1 y_1 + u_2 y_2$, where $y_c = y_1 + y_2$

Looking for $u_1', u_2'$ $\to$ $y_1 u_1' + y_2 u_2' = 0$
$y_1' u_1' + y_2' u_2' = f(t)$

$A = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$

OPTIONAL

$, \bar{u}' = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}, \bar{f} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$

$A\bar{u}' = \bar{f} \to$

$\bar{u}' = A^{-1}\bar{f}$

$= \frac{1}{|A|}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}\begin{bmatrix} 0 \\ f(t) \end{bmatrix}$

$= \frac{1}{|A|}\begin{bmatrix} -y_2 f(t) \\ y_1 f(t) \end{bmatrix}$

$= \frac{1}{|A|}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \frac{1}{w}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}$

$= \frac{1}{y_1 y_2' - y_2 y_1'}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}$

Wronskian

$w = |A| = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' = w$

$w_1 = \begin{vmatrix} 0 & y_2 \\ f(t) & y_2' \end{vmatrix} = -y_2 f(t)$

$w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(t) \end{vmatrix} = y_1 f(t)$

$u_1' = \frac{w_1}{w} = \frac{-y_2 f(t)}{y_1 y_2' - y_2 y_1'}$

$u_2' = \frac{w_2}{w} = \frac{y_1 f(t)}{y_1 y_2' - y_2 y_1'}$ $y_1(x)\int \frac{w_1}{w} + y_2(x)\int \frac{w_2}{w}$

Then $y_p = u_1 y_1 + u_2 y_2 = y_1(x)\int_0^x \frac{-y_2(t) f(t)}{y_1' y_2 - y_2' y_1} dt + y_2(x)\int_0^x \frac{y_1(t) f(t)}{y_1' y_2 - y_2' y_1} dt$

$= \int_0^t \frac{(y_2(x) y_1(t) - y_1(x) y_2(t))}{y_1' y_2 - y_2' y_1} f(t)\, dt$

$G(x,t) = \frac{y_1(t) y_2(x) - y_1(x) y(t)}{y_1'(t) y_2(t) - y_2'(t) y_1(t)}$

**OPTIONAL**

Cramer's Rule 2 vars

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Pf: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$

$$= \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} \frac{ad-bc}{ad-bc} & 0 \\ 0 & \frac{ad-bc}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So

$A\bar{x} = \bar{b} \longrightarrow$

$A^{-1}(A\bar{x}) = A^{-1}\bar{b}$

$= (A^{-1}A)\bar{x} = A^{-1}\bar{b}$

$= I\bar{x} = A^{-1}\bar{b}$

$= \bar{x} = A^{-1}\bar{b}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1+2x_2 \\ 3x_1+4x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$x_1+2x_2=5$

$3x_1+4x_2=6$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \longrightarrow$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\ \bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ \bar{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \Longrightarrow$$

$$A^{-1} = \frac{1}{4-6}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \longrightarrow$$

$$\bar{x} = A^{-1}\bar{b} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 20-12 \\ -15+6 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \end{bmatrix}$$

$$Ly = y'' - 4y' + 3y = te^t, \quad \text{s.t. } y(0) = y'(0) = 2$$

Frobenius

$$y = \sum_{k=0}^{\infty} c_k x^k$$

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1} = \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \longrightarrow xe^x = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!} =$$

Sub in:

$$\sum_{k=0}^{\infty} \Big((k+2)(k+1) c_{k+2} - 4(k+1) c_{k+1} + 3c_k\Big) x^k = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!} = \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!}$$

$$2c_2 - 4c_1 + 3c_0 + \sum_{k=1}^{\infty} \Big((k+2)(k+1) c_{k+2} - 4(k+1) c_{k+1} + 3c_k\Big) x^k = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^k$$

$$= 2c_2 - 4c_1 + 3c_0 + \sum_{k=1}^{\infty} \Big((k+2)(k+1) c_{k+2} - 4(k+1) c_{k+1} + 3c_k\Big) x^k$$

$$\Rightarrow \quad 2c_2 - 4c_1 + 3c_0 + \sum_{k=1}^{\infty} \Big((k+2)(k+1) c_{k+2} - 4(k+1) c_{k+1} + 3c_k - \frac{1}{(k-1)!}\Big) x^k$$

970-290-0550

$$\boxed{c_0 = y(0) = 2 \quad y'(0) = 2 = c_1}$$

$$2c_2 - 4(2) + 3(2) = 0$$

$$2c_2 = \boxed{c_2 = 1}$$

$$(k+2)(k+1) c_{k+2} - 4(k+1) c_{k+1} + 3c_k - \frac{1}{(k-1)!}\Big) x^k$$

$$=$$

$k=1$: $3(2)c_3 - 4(2)c_2 + 3c_1 - \frac{1}{0!}$

$$= 6c_3 - 8c_2 + 3c_1 - 1$$

$$= 6c_3 - 8(1) + 3(2) - 1$$

$$= 6c_3 - 3 = 0 \Rightarrow$$

$$c_3 = \frac{3}{6} = \boxed{\frac{1}{2} = c_3}$$

$k=2$: $4(3)c_4 - 4(3)c_3 + 3c_2 - \frac{1}{1!}$

$$= 12c_4 - 12\left(\tfrac{1}{2}\right) + 3(1) - 1$$

$$= 12c_4 - 6 + 3 - 1 = 12c_4 - 4 = 0 \Rightarrow$$

$$c_4 = \frac{4}{12} = \boxed{\frac{1}{3} = c_4}$$

$k=3$: $5(4)c_5 - 4(4)c_4 + 3c_3 - \frac{1}{2!}$

$$= 20c_5 - 16c_4 + 3c_3 - \frac{1}{2}$$

$$= 20c_5 - \frac{16}{3} + \frac{3}{2} - \frac{1}{2}$$

$$= 20c_5 + \frac{-16+3}{3} = 0 \Rightarrow$$

$$20c_5 = \frac{13}{3} \Rightarrow$$

$$\boxed{c_5 = \frac{13}{60}}$$

$$y(t) = 2 + 2t + t^2 + \frac{1}{2}t^3 + \frac{1}{3}t^4 + \frac{13}{60}t^5 + \frac{7}{60}t^6 + \frac{263}{5040}t^7 + \frac{403}{20160}t^8 + \frac{407}{60480}t^9 + \ldots$$

**Laplace Transform Method**

$$\mathcal{L}(y(t)) = F(s)$$
$$Y$$

$$Ly = y'' - 4y' + 3y = te^t, \quad y(0) = y'(0) = 2$$

$$\mathcal{L}(y'') - 4\mathcal{L}(y') + 3\mathcal{L}(y) = \mathcal{L}(te^t)$$

$$s^2F(s) - sy(0) - y'(0) - 4sF(s) - y(0) + 3F(s) = (-1)\frac{d}{ds}\mathcal{L}(e^t)$$

$$s^2F(s) - 2s - 2 - 4sF(s) - 2 + 3F(s) = -\frac{1}{s-1}$$

$$(s^2 - 4s + 3)F(s) = -\frac{1}{s-1} + 2s + 6$$

$$F(s) = \frac{-1}{(s-1)(s-3)(s-1)} + \frac{2s}{(s-1)(s-3)} + \frac{6}{(s-1)(s-3)}$$

$$= \frac{15}{8(s-1)} - \frac{1}{2(s-1)^3} - \frac{1}{4(s-1)^2} + \frac{1}{8(s-3)}$$

$$\Rightarrow y = \mathcal{L}^{-1}(\text{ABOVE})$$

$$= \frac{15}{8}e^t - \mathcal{L}^{-1}\left(\frac{1}{2}(F(s-1))^3\right) - \frac{1}{4}\mathcal{L}^{-1}\left((F(s-1))^2\right) + \frac{1}{8}e^{3t}$$

$$= \frac{15}{8}e^t + \frac{1}{8}e^{3t} - \frac{1}{2}e^t\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) - \frac{1}{4}e^t\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)$$

$$\mathcal{L}^{-1}(F(s-2))$$
$$= e^{2t}\mathcal{L}^{-1}(F(s^3))$$
$$= e^t\mathcal{L}^{-1}\left(\frac{1}{s^3}\right)$$

$$= \boxed{\frac{15}{8}e^t + \frac{1}{8}e^{3t} - \frac{1}{2}e^t \cdot \frac{t^2}{2} - \frac{1}{4}e^t \cdot t}$$

$$\boxed{= \frac{15}{8}e^t - \frac{1}{4}te^t - \frac{1}{4}t^2e^t + \frac{1}{8}e^{3t}}$$

**Look for an LRC circuit similar to the homework.**

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \mathcal{L}\left(\int_0^t f(t)\cdot 1\,d\tau\right)$$

$$= \mathcal{L}(f(t))\mathcal{L}(1) = \frac{1}{s}F(s)$$