7.5 - The Dirac Delta Function

we build a generalized function $\delta(t) \ni \mathcal{L}(\delta(t)) = 1$

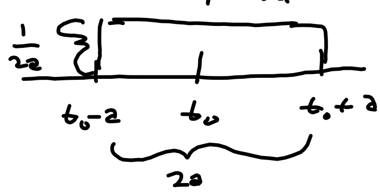
Now, T7.1.3 says $\nexists$ f that's piecws-cont. & of Exponential Order that has $\mathcal{L}(f(t)) = 1 = F(s) \xrightarrow{s\to\infty} 1$

Unit Impulse : $\delta_a(t-t_0) = \begin{cases} 0 & \text{if } t \in [0, t_0 - a) \\ \frac{1}{2a} & \text{if } t \in [t_0 - a, t_0 + a) \\ 0 & \text{if } t \geq t_0 + a \end{cases}$

Note: $\int_0^\infty \delta_a(t-t_0)\,dt = \int_{t_0-a}^{t_0+a} \frac{1}{2a}\,dt = \frac{1}{2a}\left(t_0 + a - (t_0 - a)\right) = 1$

For this reason, we call $\delta_a(t-a)$ a "unit impulse."

We define $\delta(t-t_0) = \lim_{a\to 0} \delta_a(t-t_0)$



$\boxed{\text{T7.5.1}}$ $\mathcal{L}(\delta_a(t-t_0)) = e^{-st_0}$

$\boxed{\text{Pf}}$ $\mathcal{L}(\delta_a(t-t_0)) = \mathcal{L}\left(\frac{1}{2a}\left[\mathcal{U}(t-(t_0-a)) - \mathcal{U}(t-(t_0+a))\right]\right)$

$= \frac{1}{2a}\left[\frac{e^{-s(t_0-a)}}{s} - \frac{e^{-s(t_0+a)}}{s}\right] = e^{-st_0}\left[\frac{e^{sa}-e^{-sa}}{2as}\right]$

$\rightarrow \mathcal{L}(\delta(t-t_0)) = \lim_{a\to 0} \mathcal{L}\left(e^{-st}\left(\frac{e^{sa}-e^{-sa}}{2as}\right)\right)$

$= \mathcal{L}\left(e^{-st_0} \lim_{a\to 0}\left[\frac{e^{sa}-e^{-sa}}{2as}\right]\right) = e^{-st_0}\left[\lim_{a\to 0}\left(\frac{se^{sa}+se^{-sa}}{2s}\right)\right]$

$= e^{-st_0}\left[\frac{se^0+se^0}{2s}\right] = e^{-st_0}$ ∎

$\boxed{E1}$ Solve $y'' + y = 4\delta(t-2\pi)$ s.t.

(a) $y(0)=1, y'(0)=0$ (b) $y(0)=0, y'(0)=0$

In both (a) and (b), the mass hanging from a spring is given a sharp blow at *t* = 2Pi. In (a), the object is one unit below (stretching the spring) equilibrium position. In (b), the mass is at rest in equilibrium position.

(a) $\mathcal{L}(y''+y) = s^2\mathcal{L}(y) - sy(0) - y'(0) + \mathcal{L}(y(t)) = 4e^{-2\pi s}$

$\Rightarrow s^2\mathcal{L}(y) - sy(0) - y'(0) + \mathcal{L}(y) = 4e^{-2\pi s}$

$\Rightarrow (s^2+1)\mathcal{L}(y(t)) = 4e^{-2\pi s} + s \longrightarrow$

$\mathcal{L}(y(t)) = \dfrac{4e^{-2\pi s}}{s^2+1} + \dfrac{s}{s^2+1}$

$\Rightarrow y(t) = 4\sin(t-2\pi)\mathcal{U}(t-2\pi) + \cos(t)$

$$= \begin{cases} \cos(t) & 0 \le t < 2\pi \\ \cos(t) + 4\sin(t) & 2\pi \le t \end{cases}$$

$(\sin(t-2\pi) = \sin(t))$

(b) $y(0)=0, y'(0)=0$

$\Rightarrow s^2\mathcal{L}(y) - sy(0) - y'(0) + \mathcal{L}(y) = 4e^{-2\pi s}$

$\Rightarrow s^2\mathcal{L}(y) + \mathcal{L}(y) = 4e^{-2\pi s}$

$\Rightarrow (s^2+1)\mathcal{L}(y) = 4e^{-2\pi s}$

$\Rightarrow \mathcal{L}(y) = 4e^{-2\pi s}\left(\dfrac{1}{s^2+1}\right)$

$\Rightarrow y(t) = 4\sin(t-2\pi)\mathcal{U}(t-2\pi)$

$$= \begin{cases} 0, & 0 \le t < 1 \\ 4\sin(t), & 1 \le t \end{cases}$$

$\uparrow$

$\sin(t-2\pi) = \sin(t)$

Alternate Def'n

If f is cont^s, then $\int_0^\infty f(t)\delta(t-t_0)\,dt = f(t_0)$