Section 7.4 - Operational Properties II

$\boxed{7.4.1}$ Derivatives of a Transform

<u>ASSUME</u> $\frac{d}{ds}\int f(t,s)\,dt = \int \frac{d}{ds}[f(t,s)]\,dt$

We're not going to be rigorous about proving this

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty \frac{d}{ds}[e^{-st}f(t)]\,dt$$

$$= \int_0^\infty -te^{-st}f(t)\,dt = -\int_0^\infty e^{-st}(tf(t))\,dt = -\mathcal{L}(tf(t))$$

That is $\boxed{\mathcal{L}(tf(t)) = -\frac{d}{ds}\mathcal{L}(f(t))}$

$\boxed{T7.4.1}$ If the function f is piecewise continuous on $[0, \infty)$ and of exponential order with c as specified in Definition 7.1.2 and $\mathscr{L}\{f(t)\} = F(s)$, then for $n = 1, 2, 3, \ldots$ and $s > c$,

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s).$$

INVERSE VERSION: $\mathcal{L}^{-1}\left(\frac{d^n}{ds^n}(F(s))\right) = (-1)^n t^n f(t)$

E2 Solve $y'' + 16y = \cos(4t)$, $y(0)=0$, $y'(0)=1$

$s^2\mathcal{L}(y) - sy(0) - y'(0) + 16\mathcal{L}(y) = \mathcal{L}(\cos(4t)) = \frac{s}{s^2+16}$

$s^2\mathcal{L}(y) - 0 - 1 + 16\mathcal{L}(y) = \frac{s}{s^2+16}$

$(s^2+16)\mathcal{L}(y) = 1 + \frac{s}{s^2+16}$

$\mathcal{L}(y) = \frac{1}{4}\frac{4}{s^2+16} + \frac{s}{(s^2+16)^2} \longrightarrow$

$y = \frac{1}{4}\sin(4t) + \underbrace{\mathcal{L}^{-1}\left(\frac{s}{(s^2+16)^2}\right)} = \sin(4t) + \mathcal{L}^{-1}\left(\frac{d}{ds}\left[\frac{1}{2}\frac{1}{s^2+16}\right]\right)$

Try to apply the theorem by recognizing this as $\frac{d}{ds}$[something]

$\int \frac{s}{(s^2+16)^2}ds = \frac{1}{2}\int (s^2+16)^{-2}(2s\,ds) = \frac{1}{2}\left(-1\,(s^2+16)^{-1}\right)$

So, if $F(s) = \frac{s}{s^2+16}$, then $-\frac{d}{ds}[F(s)] = -\left(-\frac{1}{2}(s^2+16)^{-1}\right)$

$= \frac{1}{2}\left(\frac{1}{s^2+16}\right) = \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{4}{s^2+16}\right) = \frac{1}{8}\left(\frac{4}{s^2+16}\right)$

So $\mathcal{L}^{-1}\left(\frac{s}{s^2+16}\right) = \frac{1}{8}t\,\mathcal{L}^{-1}\left(\frac{4}{s^2+16}\right) = \frac{1}{8}t\sin(4t)$

$\rightarrow \boxed{y = \frac{1}{4}\sin(4t) + \frac{1}{8}t\sin(4t)}$

This can be handy for dealing with $\mathcal{L}(t^n f(t))$

7.4.2 Transforms of Integrals

CONVOLUTION:

If f, g cnts on $[0, \infty)$, then the CONVOLUTION of f and g is given by

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = (f*g)(t)$$

$\boxed{E4}$ $e^t * \sin(t) = \int_0^t e^\tau \sin(t-\tau)\, d\tau = \left[\frac{1}{2} e^t \left(\cos(t-\tau) + \sin(t-\tau)\right)\right]_0^t$

By parts

$$= \boxed{-\frac{1}{2}\cos(t) - \frac{1}{2}\sin(t) + \frac{1}{2} e^t}$$

and

$$\mathcal{L}\left(e^t * \sin(t)\right) = \mathcal{L}\left(-\frac{1}{2}\cos(t) - \frac{1}{2}\sin(t) + \frac{1}{2} e^t\right)$$

$$= -\frac{1}{2}\left(\frac{s}{s^2+1}\right) - \frac{1}{2}\left(\frac{1}{s^2+1}\right) + \frac{1}{2}\left(\frac{1}{s-1}\right)$$

$$\int_0^t e^{\tau}\sin(t-\tau)\,d\tau$$

Just because we *can* integrate by parts doesn't mean we *want* to.

$dv = e^{\tau}d\tau \qquad u = \sin(t-\tau)$

$v = e^{\tau} \qquad du = \cos(t-\tau)(-1)\,d\tau$

$$uv - \int v\,du = e^{\tau}\sin(t-\tau)\Big]_0^t - \int_0^t e^{\tau}(-\cos(t-\tau))\,d\tau$$

$u = \cos(t-\tau) \qquad dv = e^{\tau}d\tau$

$du = -\sin(t-\tau)(-1)\,d\tau \qquad v = e^{\tau}$

$= \sin(t-\tau)\,d\tau$

$$= e^{t}\sin(t-t) - e^{0}\sin(t) + e^{\tau}\cos(t-\tau)\Big]_0^t - \int_0^t e^{\tau}\sin(t-\tau)\,d\tau$$

$$= -\sin(t) + e^{t}\cos(t-t) - e^{0}\cos(t) - \int_0^t e^{\tau}\sin(t-\tau)\,d\tau$$

$$= -\sin(t) + e^{t} - \cos(t) - \int_0^t e^{\tau}\sin(t-\tau)\,d\tau \longrightarrow$$

$$2\int_0^t e^{\tau}\sin(t-\tau)\,dt = e^{t} - \sin(t) - \cos(t)$$

$$\longrightarrow \int_0^t e^{\tau}\sin(\tau-\tau)\,d\tau = \frac{1}{2}\left[e^{t} - \sin(t) - \cos(t)\right] \checkmark$$

The convolution is "interpreted as a GENERALIZED PRODUCT of f & g.

$f*g = g*f$ Commutative

$f*(g*h) = (f*g)*h$ Associative

$f*(g+h) = f*g + f*h$ Distributive

$(cf)*g = c(f*g)$ Scalar Multiplication.

$\boxed{T\ 7.4.2}$ f & g pcws-cnts on $[0,\infty)$, of E.O. (with constant c)

$\Rightarrow \mathcal{L}(f*g) = \mathcal{L}(f(t))\mathcal{L}(g(t))$ OR $F(s)G(s)$

Proof beyond the scope of this course (Calc III required)

$\boxed{E5}$ $\mathcal{L}\left(\int_0^t e^{\tau}\sin(t-\tau)\,d\tau\right) = \mathcal{L}(e^t * \sin(t)) = \mathcal{L}(e^t)\mathcal{L}(\sin(t))$

$= \left(\frac{1}{s-1}\right)\left(\frac{1}{s^2+1}\right)$

<u>INVERSE FORM of T 7.4.2!</u>

$\mathcal{L}^{-1}\left(F(s)G(s)\right) = f(t)*g(t)$

E6 Much better way to do

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+k^2)^2}\right) = \mathcal{L}^{-1}\left(\left(\frac{1}{s^2+k^2}\right)\left(\frac{1}{s^2+k^2}\right)\right)$$

If you can recognize a transform as the derivative of something, you can use Theorem 7.4.1.

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2+k^2}\right) * \mathcal{L}^{-1}\left(\frac{1}{s^2+k^2}\right)$$

$$= \frac{1}{k}\mathcal{L}^{-1}\left(\frac{k}{s^2+k^2}\right) * \frac{1}{k}\mathcal{L}^{-1}\left(\frac{k}{s^2+k^2}\right)$$

$$= \frac{1}{k^2}\left(\sin(kt) * \sin(kt)\right)$$

$$= \frac{1}{k^2}\int_0^t \sin(k\tau)\sin(k(t-\tau))\,d\tau = -\frac{-\sin(t)\,k+\sin(k\,t)}{k^2\,(k-1)\,(k+1)}$$

$$= \frac{1}{k^2}\int_0^t \frac{1}{2}\left[\cos(k\tau-(kt-k\tau)) - \cos(k\tau+(kt-k\tau))\right]d\tau$$

$$= \frac{1}{2k^2}\int_0^t \left(\cos(2k\tau-kt) - \cos(kt)\right)d\tau$$

$$= \frac{1}{2k^2}\left[\frac{1}{2k}\sin(2k\tau-kt)\right]_0^t - \frac{1}{2k^2}\left[\tau\cos(kt)\right]_0^t$$

$$= \frac{1}{4k^3}\left[\sin(2kt-kt) - \sin(-kt)\right] - \frac{1}{2k^2}\left[t\cos(kt) - 0\cdot\cos(0)\right]$$

$$= \frac{1}{4k^3}\left[\sin(kt)+\sin(kt)\right] - \frac{1}{2k^2}\left[t\cos(kt)\right]$$

$$= \frac{1}{2k^3}\sin(kt) - \frac{1}{2k^2}t\cos(kt)$$

Transform of an Integral $g(t)=1 \Rightarrow G(s)=\frac{1}{s}$ so

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \mathcal{L}\left(\int_0^t f(\tau)\cdot 1\, d\tau\right) \quad (g(t-\tau)=1)$$

$$= F(s)G(s) = F(s)\cdot\frac{1}{s} = \frac{F(s)}{s}.$$

Summarize:

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{F(s)}{s}$$

Inverse Form: $\int_0^t f(\tau)d\tau = \mathcal{L}^{-1}\left(\frac{F(s)}{s}\right)$

VOLTERRA INTEGRAL EQUATION We solve:

$$f(t) = g(t) + \int_0^t f(\tau)h(t-\tau)d\tau \qquad (10)$$

$g(t), h(t)$ given. Find $f(t)$.

$\boxed{E7}$ Solve $f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}d\tau$ for $f(t)$

SEE Next page.

$$\mathcal{L}(f(t)) = \mathcal{L}(3t^2 - e^{-t}) - \mathcal{L}(f(\tau)e^{t-\tau}d\tau)$$

$$= 3\mathcal{L}(t^2) - \mathcal{L}(e^{-t}) - \mathcal{L}(f(t))\mathcal{L}(e^t)$$

$$\Rightarrow \mathcal{L}(f(t)) + \mathcal{L}(f(t))\mathcal{L}(e^t) = 3\left(\frac{2}{s^3}\right) - \frac{1}{s+1} \longrightarrow$$

$$\mathcal{L}(f(t))(1+\mathcal{L}(e^t)) = \mathcal{L}(f(t))\left(1+\frac{1}{s-1}\right) = \frac{6}{s^3} - \frac{1}{s+1} \longrightarrow$$

$$\mathcal{L}(f(t)) = \frac{\frac{6}{s^3} - \frac{1}{s+1}}{\frac{s-1+1}{s-1}} = \left(\frac{6}{s^3} - \frac{1}{s+1}\right)\left(\frac{s-1}{s}\right) = \frac{6s-6}{s^4} - \frac{s-1}{s(s+1)}$$

$$= \frac{6}{s^3} - \frac{6}{s^4} - \frac{s}{s(s+1)} + \frac{1}{s(s+1)}$$

$$= \frac{6}{s^3} - \frac{6}{s^4} - \frac{1}{s+1} - \frac{1}{s(s+1)} \longrightarrow$$

$$f(t) = \mathcal{L}^{-1}\left(3\left(\frac{2}{s^3}\right)\right) - \mathcal{L}^{-1}\left(\frac{3!}{s^4}\right) - e^{-t} - \mathcal{L}^{-1}\left(\frac{1}{s}\cdot\frac{1}{s+1}\right)$$

$$= 3t^2 - t^3 - e^{-t} - \mathcal{L}^{-1}\left(\frac{1}{s}\right) * \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$- 1 * e^{-t}$$

$$= 3t^2 - t^3 - e^{-t} - \int_0^t e^{-\tau}d\tau = 3t^2 - t^3 - e^{-t} + \int_0^t e^{-\tau}(-d\tau)$$

$$= 3t^2 - t^3 - e^{-t}$$

$\boxed{E7}$ Solve $f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}\,d\tau$ for $f(t)$

$$\mathcal{L}(f(t)) = \mathcal{L}(3t^2 - e^{-t}) - \mathcal{L}\left(\int_0^t f(\tau)e^{t-\tau}\,d\tau\right)$$

$$= 3\mathcal{L}(t^2) - \mathcal{L}(e^{-t}) - \mathcal{L}(f(t))\mathcal{L}(e^{t})$$

$$= 3\left(\frac{2}{s^3}\right) - \frac{1}{s+1} - \mathcal{L}(f(t))\left(\frac{1}{s-1}\right) \longrightarrow$$

$$\mathcal{L}(f(t)) + \mathcal{L}(f(t))\left(\frac{1}{s-1}\right) = \frac{6}{s^3} - \frac{1}{s+1} \longrightarrow$$

$$\mathcal{L}(f(t))\left[1 + \frac{1}{s-1}\right] = \mathcal{L}(f(t))\left(\frac{s-1+1}{s-1}\right) = \mathcal{L}(f(t))\left(\frac{s}{s-1}\right) = \frac{6}{s^3} - \frac{1}{s+1} \longrightarrow$$

$$\mathcal{L}(f(t)) = \frac{6}{s^3}\left(\frac{s-1}{s}\right) - \frac{1}{s+1}\left(\frac{s-1}{s}\right) =$$

$$= \frac{6s-6}{s^4} - \frac{s-1}{s(s+1)} = \frac{6}{s^3} - \frac{6}{s^4} - \frac{1}{s+1} + \frac{1}{s(s+1)}$$

$$\Longrightarrow f(t) = 3t^2 - t^3 - e^{-t} + \mathcal{L}^{-1}\left(\frac{1}{s}\right) * \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) \quad \text{— T7.4.2}$$

$$= 3t^2 - t^3 - e^{-t} + \int_0^t e^{-\tau}\,d\tau$$

$$= 3t^2 - t^3 - e^{-t} - e^{-t} + 1$$

$$\boxed{f(t) = 3t^2 - t^3 - 2e^{-t} + 1}$$

SERIES CIRCUITS

$E(t) = \sum(\text{voltage Drops})$

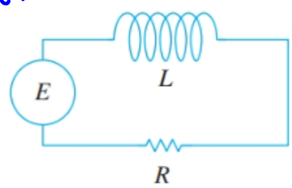
$= L\frac{di}{dt} + Ri(t) + \frac{1}{C}q(t)$, where

i = current, q = charge on capacitor,

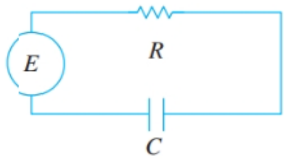
L = inductance (henries)

R = Resistance (ohms)

C = Capacitance of capacitor. (Farads)



(a) *LR*-series circuit



(b) *RC*-series circuit

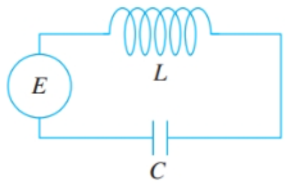
LR-circuit:

$$L\frac{di}{dt} + Ri = E(t)$$

RC-circuit:

$$Ri + \frac{1}{C}q = E(t)$$

$$i = \frac{dq}{dt} \Longrightarrow Ri + \frac{1}{C}\int_0^t i(\tau)\,d\tau$$



(c) *LC*-series circuit

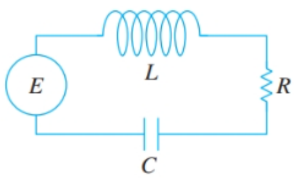
LC circuit:

$$L\frac{di}{dt} + \frac{1}{C}q = L\frac{di}{dt} + \frac{1}{C}\int_0^t i(\tau)\,d\tau = E(t)$$

Integro-Differential Equation.

LRC:

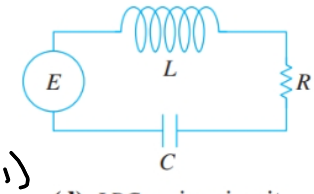
$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\tau)\,d\tau = E(t)$$



(d) *LRC*-series circuit

$\boxed{E8}$ Find $i(t)$ in LRC-circuit

$L=.1h$, $R=2\Omega$, $C=0.1f$, $i(0)=0$, & $E(t)=120t-120t\mathcal{U}(t-1)$



(d) *LRC*-series circuit

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int_0^t i(\tau)\,d\tau = E(t)$$

$$.1\frac{di}{dt}+2i+\frac{1}{.1}\int_0^t i(\tau)\,d\tau = 120t-120t\mathcal{U}(t-1)$$

$$\frac{1}{10}(sI(s)-i(0))+2I(s)+10\mathcal{L}\left(\int_0^t i(\tau)\,d\tau\right) = 120\mathcal{L}(t)-\mathcal{L}(120t\mathcal{U}(t-1))$$

$$\frac{1}{10}sI(s)+2I(s)+10\mathcal{L}(i(t))\mathcal{L}(1) = 120\left(\frac{1}{s^2}\right)-120e^{-as}(t+1)$$

$$\frac{1}{10}sI(s)+2I(s)+10\cdot\frac{I(s)}{s} = \frac{120}{s^2}-120e^{-s}\mathcal{L}(t+1) \qquad \mathcal{L}(g(t)\mathcal{U}(t-a)) \Rightarrow e^{-as}g(t+a)$$

$$I(s)\left(\frac{s}{10}-2+\frac{10}{s}\right) = \frac{120}{s^2}-120e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)$$

$$I(s)\left(\frac{s^2+20s+100}{10s}\right) = \frac{120}{s^2}-120e^{-s}\left(\frac{1}{s^2}\right)-120e^{-s}\left(\frac{1}{s}\right)$$

$$I(s)=\frac{10s}{s^2+20s+100}-120e^{-s}\left(\frac{1}{s^2}\right)\left(\frac{10s}{s^2+20s+100}\right)-120e^{-s}\left(\frac{1}{s}\right)\left(\frac{10s}{s^2+20s+100}\right)$$

$$\left(\begin{array}{l} s^2+20s+10^2-10^2+100 \\ =(s+10)^2 \end{array}\right)$$

$$=\frac{10s}{(s+10)^2}-120e^{-s}\left(\frac{10}{s(s+10)^2}\right)-120\left(\frac{10e^{-s}}{(s+10)^2}\right)$$

Maple Says:

$$\mathcal{L}(i(t))=I(s)= -\frac{1200\,(e^{-s}s+e^{-s}-1)}{s\,(s^2+20s+100)} \Longrightarrow$$

$i(t)=$ 12 Heaviside$(1-t)$ $-$ 12 Heaviside$(t-1)\,e^{-10t+10}(90t-91)-12e^{-10t}(10t+1)$

$$= \begin{cases} -12e^{-10t}(10t+1)+12 & t<1 \\ -132e^{-10}+24 & t=1 \\ -12e^{-10t}(10t+1)-12e^{-10t+10}(90t-91) & 1<t \end{cases}$$

$$.1\frac{di}{dt}+2i+\frac{1}{.1}\int_0^t i(\tau)\,d\tau = 120t-120t\,\mathcal{U}(t-1)$$

$$\frac{1}{10}\mathcal{L}\left(\frac{di}{dt}\right)+2\mathcal{L}(i)+10\mathcal{L}\left(\int_0^t i(\tau)\cdot 1\,d\tau\right)=120\left(\frac{1}{s^2}\right)-\mathcal{L}\left(120t\,\mathcal{U}(t-1)\right)$$

$$\frac{1}{10}\left(sI(s)-i(0)\right)+2I(s)+10\,I(s)\left(\frac{1}{s}\right)=\frac{120}{s^2}-120\mathcal{L}(t+1)\,e^{-s}$$

$$\frac{s}{10}I(s)+2I(s)+\frac{10}{s}I(s)=\frac{120}{s^2}-120e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)$$

$$\left(\frac{s}{10}+2+\frac{10}{s}\right)I(s)=\frac{120}{s^2}-120e^{-s}\cdot\frac{1}{s^2}-120e^{-s}\cdot\frac{1}{s}$$

$$\frac{s^2+20s+100}{10s}I(s)=\frac{120}{s^2}-120e^{-s}\cdot\frac{1}{s^2}-120e^{-s}\cdot\frac{1}{s}$$

$$I(s)=\frac{10s}{(s+10)^2}\cdot\frac{120}{s^2}-\frac{10s}{(s+10)^2}\cdot\frac{120e^{-s}}{s^2}-\frac{10s}{(s+10)^2}\cdot\frac{120e^{-s}}{s}$$

$$I(s)=\frac{1200}{s(s+10)^2}-\frac{1200e^{-s}}{s(s+10)^2}-\frac{1200e^{-s}}{(s+10)^2}$$

$$\frac{1}{s(s+10)^2}=\frac{A}{s}+\frac{B}{s+10}+\frac{C}{(s+10)^2}=\frac{\frac{1}{100}}{s}-\frac{\frac{1}{100}}{s+10}-\frac{\frac{1}{10}}{(s+10)^2}$$

$$1=A(s+10)^2+Bs(s+10)+Cs$$

$$=As^2+20As+100A+Bs^2+10Bs+Cs$$

s^2: $A+B=0$

s: $20A+10B+C=0$

1: $100A=1 \rightarrow \boxed{A=\frac{1}{100}} \rightarrow \boxed{B=-\frac{1}{100}} \rightarrow$

$$20\left(\frac{1}{100}\right)+10\left(-\frac{1}{100}\right)+C=0$$

$$\frac{20}{100}-\frac{10}{100}+C=0$$

$$20-10+100C=0$$

$$100C=-10$$

$$\boxed{C=-\frac{1}{10}}$$

$$I(s) = \frac{1200}{s(s+10)^2} - \frac{1200e^{-s}}{s(s+10)^2} - \frac{1200e^{-s}}{(s+10)^2}$$

lost this.
$\mathcal{L}^{-1}(\text{this})$ is
$-1200(t-1)e^{-10(t-1)}$

$$= 1200\left[\frac{1}{100}\left(\frac{1}{s}\right) - \frac{1}{100}\left(\frac{1}{s+10}\right) - \frac{1}{10}\left(\frac{1}{(s+10)^2}\right) - e^{-s}\left[\frac{1}{100}\left(\frac{1}{s}\right) - \frac{1}{100}\left(\frac{1}{s+10}\right) - \frac{1}{10}\left(\frac{1}{(s+10)^2}\right)\right]\right]$$

$$\rightarrow i(t) = 1200\left[\frac{1}{100}(1) - \frac{1}{100}e^{-10t} - \frac{1}{10}\mathcal{L}^{-1}\left(\frac{1}{(s+10)^2}\right) - \left[\underbrace{\frac{1}{100}(1) - \frac{1}{100}e^{-10(t-1)} - \frac{1}{10}\left(te^{-10(t-1)}\right)}_{\text{time } \mathcal{U}(t-1)}\right]\right]$$

$$= 12 - 12e^{-10t} - 120te^{-10t} - \left[12 - 12e^{-10(t-1)} - 120te^{-10(t-1)}\right]\mathcal{U}(t-1)$$

$$= 12 - 12\mathcal{U}(t-1) - 12e^{-10t} + 12e^{-10(t-1)}\mathcal{U}(t-1) - 120te^{-10t} + 120(t-1)e^{-10(t-1)}\mathcal{U}(t-1)$$

$$= 12(1-\mathcal{U}(t-1)) - 12\left(e^{-10t} - e^{-10(t-1)}\mathcal{U}(t-1)\right) - 120te^{-10t} + 120(t-1)e^{-10(t-1)}\mathcal{U}(t-1)$$

$$= 12(1-\mathcal{U}(t-1)) - 12\left(e^{-10t} - e^{-10(t-1)}\mathcal{U}(t-1)\right) - 120\left(te^{-10t} - \left(te^{-10(t-1)} + e^{-10(t-1)}\right)\mathcal{U}(t-1)\right)$$

$$- 1200(t-1)e^{-10(t-1)}\mathcal{U}(t-1)$$

$= 12(1-\mathcal{U}(t-1))$ .... **

#12 $f(t) = \cos(t) + \int_0^t e^{-\tau} f(t-\tau)\,d\tau$

$\Rightarrow F(s) = \mathcal{L}(\cos(t)) + \mathcal{L}(e^{-t} * f(t))$

$= \frac{s}{s^2+1} + \left(\frac{1}{s+1}\right) F(s) \longrightarrow$

$F(s) - \frac{1}{s+1} F(s) = \frac{s+1-1}{s+1} F(s) = \frac{s}{s+1} F(s) = \frac{s}{s^2+1} \longrightarrow$

$F(s) = \frac{s+1}{s} \cdot \frac{s}{s^2+1} = \frac{s+1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2+1} \longrightarrow$

$\mathcal{L}^{-1}(LHS) = \mathcal{L}^{-1}(RHS) \longrightarrow$

$\boxed{f(t) = \cos(t) + \sin(t)}$

7.4.3 - Transform of a Periodic Function

$\boxed{T\ 7.4.3}$ f pcws-cont^s on $[0,\infty)$ of E.O. & periodic with period T.

$$\Longrightarrow \mathcal{L}(f(t)) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

Recall: $\sum_{n=3}^{\infty} f(n) = \sum_{n=0}^{\infty} f(n+3)$

$\boxed{Pf}$

$$\mathcal{L}(f(t)) = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$

Now, $\int_T^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-s(u+T)} f(u+T)\,du = e^{-sT}\int_0^\infty e^{-su} f(u)\,du$, since $f(u+T) = f(u)$.

$$= e^{-sT}\mathcal{L}(f(t)) \longrightarrow$$

This should remind you of integrating $\int e^t \sin(t)\,dt$ by parts.

$$\mathcal{L}(f(t)) = \int_0^T e^{-st} f(t)\,dt + e^{-sT}\mathcal{L}(f(t)) \longrightarrow$$

$$\mathcal{L}(f(t))(1-e^{-sT}) = \int_0^T e^{-st} f(t)\,dt \longrightarrow$$

$$\mathcal{L}(f(t)) = \frac{1}{1-e^{-sT}}\int_0^\infty e^{-st} f(t)\,dt \quad \blacksquare$$

$\boxed{E9}$

$E(t)$ = square wave of period $T=2$, given on $[0,2)$ by

$$E(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases}$$

Then $\mathcal{L}(E(t)) = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} E(t)\,dt = \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}\,dt + \int_1^2 e^{-st}\cdot 0\,dt$

* $a^2 - b^2 = (a-b)(a+b)$
$1 - e^{-2s} = 1^2 - (e^{-s})^2$

$$= \frac{1}{1-e^{-2s}}\left[-\frac{1}{s}e^{-st}\right]_0^1 = \frac{1}{1-e^{-2s}}\left[-\frac{1}{s}(e^{-s}-1)\right]$$

$$= \frac{-(e^{-s}-1)}{s(1-e^{-2s})} = \frac{1-e^{-s}}{s(1-e^{-s})(1+e^{-s})}^{*} = \boxed{\frac{1}{s(1+e^{-s})}}$$

$\boxed{E10}$ $L\frac{di}{dt} + Ri = E(t)$, where $E(t)$ is as in E9, $i(0) = 0$

$$\mathcal{L}\left(L\frac{di}{dt}\right) + R\mathcal{L}(i) = \mathcal{L}(E(t)) = \frac{1}{s(1+e^{-s})}$$

$$L\left(s\mathcal{L}(i) - i(0)\right) + R\mathcal{L}(i) = (sL+R)\mathcal{L}(i) = \frac{1}{s(1+e^{-s})}$$

$$\mathcal{L}(i) = \frac{1}{s(1+e^{-s})(sL+R)} = \frac{1}{Ls(1+e^{-s})(s+\frac{R}{L})} = \frac{\frac{1}{L}}{s(1+e^{-s})(s+\frac{R}{L})}$$

$$= \left(\frac{1}{s}\right)\left(\frac{1}{s+\frac{L}{R}}\right)\left(\frac{1}{L}\right)\left(\frac{1}{1+e^{-s}}\right) =$$

$$= \frac{\frac{1}{L}}{s(s+\frac{L}{R})}\left(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots\right)$$

(Subs e^{-s} for x in Power-Series expansion for $f(x) = \frac{1}{1+x}$.)

$$= \frac{1}{L}\left(\frac{L}{R}\cdot\frac{1}{s} - \frac{L}{R}\cdot\frac{1}{s+\frac{R}{L}}\right)\left(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots\right)$$

$$= \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right)\left(1 - e^{-s} + e^{2s} - e^{-3s} + \cdots\right)$$

$$= \frac{1}{R}\left[\frac{1}{s}\left(1 - e^{-s} + e^{-2s} \cdots\right) - \frac{1}{s+\frac{R}{L}}\left(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots\right)\right]$$

$\longrightarrow$

$$i(t) = \frac{1}{R}\left[1 - \mathcal{U}(t-1) + \mathcal{U}(t-2) - \mathcal{U}(t-3) + \cdots\right]$$

$$- \frac{1}{R}\left[e^{-\frac{R}{L}t} - e^{-\frac{R}{L}(t-1)}\mathcal{U}(t-1) + e^{-\frac{R}{L}(t-2)}\mathcal{U}(t-2) - \cdots\right]$$

$$= \boxed{\frac{1}{R}\left[1 - e^{-\frac{R}{L}t}\right] + \frac{1}{R}\sum_{n=1}^{\infty}(-1)^n\left(1 - e^{-\frac{R}{L}(t-n)}\right)\mathcal{U}(t-n)}$$

$$S = 1 + x + x^2 + x^3 + \dots + x^n + \dots$$

$$- \quad xS = \quad x + x^2 + x^3 + \dots + x^n + \dots$$

$$(1-x)S = 1 \Longrightarrow$$

Geometric.

$$S = \frac{1}{1-x}$$

$$\Longrightarrow \frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \dots$$