7.3 - Operational Properties I

Theorem 7.3.1 First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ exists for $s > c$ and a is any real number, then

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a)$$

for $s > a + c$.

$$\mathscr{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

Pf:

$$\mathscr{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-st+at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= F(s-a) \blacksquare$$

Example 1 Using the First Translation Theorem

Find **(a)** $\mathscr{L}\{e^{5t}t^3\}$ **(b)** $\mathscr{L}\{e^{-2t}\cos 4t\}$.

Solution The results follow from Theorems 7.1.1 and 7.3.1.

(a) $\mathscr{L}\{e^{5t}t^3\} = \mathscr{L}\{t^3\}\big|_{s\to s-5} = \left.\dfrac{3!}{s^4}\right|_{s\to s-5} = \dfrac{6}{(s-5)^4}$

New notation

(b) $\mathscr{L}\{e^{-2t}\cos 4t\} = \mathscr{L}\{\cos 4t\}\big|_{s\to s-(-2)} = \left.\dfrac{s}{s^2+16}\right|_{s\to s+2} = \dfrac{s+2}{(s+2)^2+16}$

I don't like the curly brackets. But it's a matter of taste. Main reason I don't like it is because I reserve {} brackets for sets.

Example 2 Partial Fractions: Repeated Linear Factors

Find (a) $\mathscr{L}^{-1}\left\{\dfrac{2s+5}{(s-3)^2}\right\}$ (b) $\mathscr{L}^{-1}\left\{\dfrac{s/2+5/3}{s^2+4s+6}\right\}$.

$$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}, \text{ etc.}$$

$$F(s) = \frac{\frac{s}{2}+\frac{5}{3}}{s^2+4s+6} = \frac{As+B}{s^2+4s+6}, \text{ etc.}$$

$$x^2+bx+c = x^2+bx+\left(\frac{b}{2}\right)^2 - \frac{b^2}{4} + \frac{4c}{4} = \left(x+\frac{b}{2}\right)^2 + \frac{4c-b^2}{4}$$

To see the "shift," complete the square

$$s^2+4s+6 = s^2+4s+2^2-4+6 = (s+2)^2+2$$

$$\frac{\frac{s}{2}+\frac{5}{2}}{(s+2)^2+2} = \frac{\frac{1}{2}(s+2)+\frac{2}{3}}{(s+2)^2+2} = \frac{1}{2}\left(\frac{s+2}{(s+2)^2+2}\right) + \frac{2}{3}\left(\frac{1}{(s+2)^2+2}\right)$$

$= \frac{1}{2}F(s+2)$

$s \mapsto s+2$

Re-write numerator

$$\frac{s}{2}+\frac{5}{3} = \frac{s+2-2}{2} + \frac{5}{3} = \frac{s+2}{2} - 1 + \frac{5}{3} = \frac{1}{2}(s+2) + \frac{2}{3}$$

$$\mathscr{L}^{-1}\left(\frac{1}{2}\left(\frac{s+2}{(s+2)^2+2}\right)\right) + \mathscr{L}^{-1}\left(\frac{2}{3}\left(\frac{1}{(s+2)^2+2}\right)\right)$$

$\mathscr{L}\left(\frac{k}{s^2+k^2}\right) = \sin(kt)$

$$= \frac{1}{2}\mathscr{L}^{-1}\left(\left.\frac{s}{s^2+2}\right|_{s\to s+2}\right) + \frac{2}{3}\mathscr{L}^{-1}\left(\left.\frac{1}{s^2+2}\right|_{s\to s+2}\right)$$

$$= e^{-2t}\cos(\sqrt{2}t) + \frac{2}{3\sqrt{2}}\mathscr{L}^{-1}\left(\left.\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right|_{s\to s+2}\right)$$

$$\boxed{= e^{-2t}\cos(\sqrt{2}t) + \frac{2\sqrt{2}}{6}e^{-2t}\sin(\sqrt{2}t)}$$

$\frac{\sqrt{2}}{3}$ OR $\frac{2}{3\sqrt{2}}$ was also OK.

Example 3 An Initial-Value Problem

Solve $y'' - 6y' + 9y = t^2e^{3t}$, $y(0) = 2$, $y'(0) = 17$.

Solution Before transforming the DE, note that its right-hand side is similar to the function in part (a) of Example 1. After using linearity, Theorem 7.3.1, and the initial conditions, we simplify and then solve for $Y(s) = \mathscr{L}\{f(t)\}$:

$$s^2Y(s) - sy(0) - y'(0) - 6\left[sY(s) - y(0)\right] + 9Y(s) = \mathscr{L}(t^2e^{3t})$$

$$\Rightarrow s^2Y(s) - s\cdot 2 - 17 - 6sY(s) + 12 + 9Y(s) = F(s)$$

$$\Rightarrow s^2Y(s) - 6sY(s) + 9Y(s) = F(s) + 2s + 5$$

$$\Rightarrow (s^2 - 6s + 9)Y(s) = F(s) + 2s + 5$$

$$\Rightarrow Y(s) = \frac{F(s) + 2s + 5}{(s-3)^2} = \frac{\frac{2}{(s-3)^3} + 2s + 5}{(s-3)^2}$$

$$\left(F(s) = \mathscr{L}(t^2e^{3t}) = \mathscr{L}(t^2)\Big|_{s \to s-3} = \frac{2}{(s-3)^3}\right) \quad \text{Scratch}$$

$$= \frac{2}{(s-3)^5} + \frac{2s+5}{(s-3)^2}$$

(See E2):

$$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$$

$$2s+5 = A(s-3) + B = As - 3A + B \longrightarrow$$

$$2s = As \rightarrow \boxed{A = 2}$$

$$\Rightarrow -3A + B = -6 + B = 5 \rightarrow \boxed{B = 11}$$

$$= \frac{2}{(s-3)^5} + \frac{2}{s-3} + \frac{11}{(s-3)^2} \longrightarrow$$

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{2}{(s-3)^5}\right) + \mathcal{L}^{-1}\left(\frac{2}{s-3}\right) + \mathcal{L}^{-1}\left(\frac{11}{(s-3)^2}\right)$$

$$\left(\begin{array}{l} \mathcal{L}(t^n) = \frac{n!}{s^{n+1}} \Longrightarrow \\ \frac{1}{n!}\mathcal{L}(t^n) = \frac{1}{s^{n+1}} \longrightarrow \\ \mathcal{L}^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{1}{n!}t^n \end{array} \right)$$

Building what I need, instead of looking it up.

$$= \frac{2}{4!}\mathcal{L}^{-1}\left(\frac{4!}{s^5}\right) + 2\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) + 11\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right)$$

$$= \frac{2}{4!}t^4 + 2\mathcal{L}^{-1}\left(\frac{1}{s}\Big|_{s \to s-3}\right) + 11\mathcal{L}^{-1}\left(\frac{1}{s^2}\Big|_{s \to s-3}\right)$$

$$= \frac{2}{4 \cdot 3 \cdot 2}t^4 + 2\left(e^{3t}\right) + 11\left(te^{3t}\right)$$

$$\boxed{= \frac{1}{12}t^4 + 2e^{3t} + 11te^{3t} = y(t)}$$

$Ly = y'' - 6y' + 9y = t^2 e^{3t}$ can be solved by 4.3 methods

$r^2 - 6r + 9 = (r-3)^2 = 0 \longrightarrow$

$y_h = c_1 e^{3t} + c_2 t e^{3t}$

y_p: Look for $Ae^{3t} + Bte^{3t} + Ct^2 e^{3t} = y_p$

But wait! y_h has $c_1 e^{3t}$ & $c_2 t e^{3t}$ in it, already!

Multiply everything by t^2:

Case ii 4.3 (TYPE II?)

$y_p = At^2 e^{3t} + Bt^3 e^{3t} + Ct^4 e^{3t}$

$\Rightarrow y_p'' - 6y_p' + 9y_p = 12\left(e^t\right)^3 t^2 C + 6\left(e^t\right)^3 tB + 2\left(e^t\right)^3 A$

$= 12e^{3t} t^2 C + 6te^{3t} B + 2Ae^{3t}$

$= 12Ct^2 e^{3t} + 6Bte^{3t} + 2Ae^{3t} = t^2 e^{3t}$

$\Rightarrow A = B = 0$ & $12C = 1 \Rightarrow C = \frac{1}{12}$

$\longrightarrow y_p = \frac{1}{12} t^4 e^{3t}$

$\boxed{\Rightarrow y = y_h + y_p = c_1 e^{3t} + c_2 t e^{3t} + \frac{1}{12} t^4 e^{3t} \text{ by } \S 4.3}$

Definition 7.3.1 Unit Step Function

The **unit step function** $\mathcal{U}(t-a)$ is defined to be

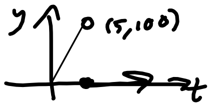
heaviside(t-a)

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a. \end{cases}$$

Example 5 A Piecewise-Defined Function

Express $f(t) = \begin{cases} 20t, & 0 \le t < 5 \\ 0, & t \ge 5 \end{cases}$ in terms of unit step functions. Graph.

$f(t) = 20t - \mathcal{U}(t-5)(20t)$



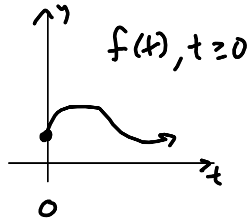
Theorem 7.3.2 Second Translation Theorem

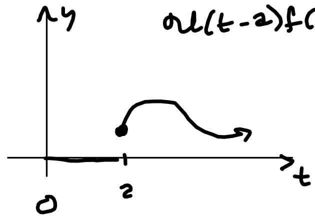
If $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > c \ge 0$, and a is a positive constant, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

for $s > c$.

$\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)\mathcal{U}(t-a)$





$\boxed{E6}$ Find $\mathcal{L}(\sin(4t)\mathcal{U}(t-\frac{\pi}{2}))$

$\left(4t = 2\pi \Rightarrow t = \frac{\pi}{2} \text{ is period} \right)$

$= \mathcal{L}(\sin(4(t-\frac{\pi}{2}))\mathcal{U}(t-\frac{\pi}{2})) = e^{-\frac{\pi}{2}s}\left(\frac{4}{s^2+16}\right)$

Inverse Form of T 7.3.2:

If $f(t) = \mathcal{L}^{-1}(F(s))$, then

$\mathcal{L}^{-1}(F(s)e^{-as}) = f(t-a)\mathcal{U}(t-a)$ (.5)

$\boxed{E8}$ $\mathcal{L}^{-1}(\frac{1}{s-4}e^{-2s}) \Rightarrow a = 2,\ F(s) = \frac{1}{s-4}$, & $\mathcal{L}^{-1}(F(s)) = e^{4t}$

$\Rightarrow \mathcal{L}^{-1}(\frac{1}{s-4}e^{-2s}) = e^{4(t-2)}\mathcal{U}(t-2)$

Algebra Trick to shoe-horn other functions:

$$t^2 = (t-2+2)^2 = (t-2)^2 + 2(2)(t-2) + 4$$
$$= (t-2)^2 + 4(t-2) + 4$$

$$\Rightarrow \mathcal{L}(t^2 \mathcal{U}(t-2)) = \mathcal{L}((t-2)^2 \mathcal{U}(t-2)) + 4\mathcal{L}((t-2)\mathcal{U}(t-2)) + 4\mathcal{L}(\mathcal{U}(t-2))$$

$$= e^{-2s}\mathcal{L}(t^2) + 4e^{-2s}\mathcal{L}(t) + 4e^{-2s}\mathcal{L}(1)$$

$$= e^{-2s}\left(\frac{2!}{s^3}\right) + 4e^{-2s}\left(\frac{1}{s^2}\right) + 4e^{-2s}\left(\frac{1}{s}\right) = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)$$

ALTERNATE Form of T7.3.2:

We don't need those tricks, if we use D7.1.1, the Defn of $\mathcal{U}(t-2)$, and the substitution $u = t-2$:

$$\mathcal{L}(g(t)\mathcal{U}(t-2)) = \int_0^\infty e^{-st} g(t)\mathcal{U}(t-2)\,dt$$

$$= \int_0^2 e^{-st} \cdot 0\,dt + \int_2^\infty e^{-st} g(t)\mathcal{U}(t-2)\,dt = \int_2^\infty e^{-st} g(t)\,dt$$

Let $u = t-2$. Then $t = u+2$ and so

$$\ldots = \int_{\substack{t=2\\u=0}}^\infty e^{-st} g(t)\,dt = \int_0^\infty e^{-s(u+2)} g(u+2)\,du = \int_0^\infty e^{-su-s2} g(u+2)\,du$$

$$= e^{-2s}\int_0^\infty e^{-su} g(u+2)\,du = e^{-2s}\mathcal{L}(g(t+2))$$

Then $\mathcal{L}(t^2\mathcal{U}(t-2)) = e^{-2s}\mathcal{L}((t+2)^2)$

$= e^{-2s}\mathcal{L}(t^2+4t+4) =$ etc. Much quicker & more direct.

It's still good to keep the algebra method previously in your bag of tricks. Completing the square is a pretty standard "trick" for working things out (recognizing patterns by re-stating in a nicer way

"It's never x^2+2x+5. It's always $(x+1)^2+4$."

Eg

$\mathcal{L}(\cos(t)\,\mathcal{U}(t-\pi)) = F(s)$

$\mathcal{L}(\cos(t+\pi-\pi)\,\mathcal{U}(t-\pi))$

$f(t) = \cos(t+\pi) \Rightarrow$

$f(t-\pi) = \cos((t-\pi)+\pi)$

$= e^{-\pi s}\mathcal{L}(\cos(t+\pi)) = e^{-\pi s}\mathcal{L}(\cos t\cos\pi - \sin t\sin\pi)$

$= e^{-\pi s}\cos\pi\,\mathcal{L}(\cos(t)) = -e^{-\pi s}\mathcal{L}(\cos(t)) = \boxed{-e^{-\pi s}\left(\frac{s}{s^2+1}\right)} = F(s)$

$\boxed{\text{E10 IVP}}$ $y'+y=f(t)$, where $f(t)=\begin{cases}0 & 0\le t<\pi\\ 3\cos t & t\ge\pi\end{cases}$,

s.t. $y(0)=5 \Longrightarrow$

$\Longrightarrow s\mathcal{L}(y)-y(0)+\mathcal{L}(y)=\mathcal{L}(f)=\mathcal{L}(3\cos(t)\,\mathcal{U}(t-\pi))$

$\Longrightarrow s\mathcal{L}(y)+\mathcal{L}(y)=e^{-\pi s}\mathcal{L}(3\cos(t+\pi))+5$

$\Longrightarrow (s+1)\mathcal{L}(y)=3e^{-\pi s}\mathcal{L}(\cos(t+\pi))+5$

$\Longrightarrow \mathcal{L}(y)=\left(\frac{1}{s+1}\right)\left(3e^{-\pi s}(-\cos(t))+5\right)$

$$=\frac{-3e^{-\pi s}\left(\frac{s}{s^2+1}\right)+5}{s+1}=\frac{3se^{-\pi s}}{(s+1)(s^2+1)}+\frac{5}{s+1}$$

$$=3e^{-\pi s}\left(\frac{s}{(s+1)(s^2+1)}\right)+\frac{5}{s+1}$$

scratch:

$$\frac{s}{(s+1)(s^2+1)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+1}$$

$s=A(s^2+1)+(s+1)(Bs+C)$

$s=-1\Longrightarrow$

$-1=A(2)$

$\boxed{A=-\frac{1}{2}}$

$s=-\frac{1}{2}s^2-\frac{1}{2}+Bs^2+Cs+Bs+C$

$\Longrightarrow$

$-\frac{1}{2}s^2+Bs^2=0\Longrightarrow$

$\boxed{B=\frac{1}{2}}$

$\Longrightarrow s=\cancel{-\frac{1}{2}s^2}-\frac{1}{2}+\cancel{\frac{1}{2}s^2}+Cs+\frac{1}{2}s+C$

$\Longrightarrow -\frac{1}{2}+C=0$ & $\Longrightarrow \boxed{C=\frac{1}{2}}$

$Cs+\frac{1}{2}s=s\Longrightarrow$

$C+\frac{1}{2}=1$

$\boxed{C=\frac{1}{2}}$ ✓

So,

$$3e^{-\pi s}\left(\frac{s}{(s+1)(s^2+1)}\right)+\frac{5}{s+1} = 3e^{-\pi s}\left(\frac{1}{2}\left(\frac{1}{s+1}\right)+\frac{1}{2}\left(\frac{s+1}{s^2+1}\right)\right)$$

$$= \frac{3}{2}e^{-\pi s}\left(\frac{-1}{s+1}+\frac{s}{s^2+1}+\frac{1}{s^2+1}\right)+\frac{5}{s+1}$$

$\Rightarrow y = \mathcal{L}^{-1}(\text{above})$

$$= \frac{3}{2}\left[\mathcal{L}^{-1}\left(e^{-\pi s}\left(\frac{1}{s+1}\right)\right)+\mathcal{L}^{-1}\left(e^{-\pi s}\left(\frac{s}{s^2+1}\right)\right)+\mathcal{L}^{-1}\left(e^{-\pi s}\left(\frac{1}{s^2+1}\right)\right)\right]+5\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

Recall:

$\mathcal{L}^{-1}(F(s)e^{-as}) = f(t-a)\mathcal{U}(t-a)$. This gives the following:

$$= \frac{3}{2}\left[e^{-(t-\pi)}\mathcal{U}(t-\pi)+\cos(t-\pi)\mathcal{U}(t-\pi)+\sin(t-\pi)\mathcal{U}(t-\pi)\right]+5e^{-t}$$

$$= \frac{3}{2}\left[e^{-(t-\pi)}-\cos(t)-\sin(t)\right]\mathcal{U}(t-\pi)+5e^{-t}$$

(scratch: $\sin(t-\pi) = \sin(t)\cos(-\pi)+\sin(-\pi)\cos(t) = -\sin(t)$)

$$= \begin{cases} 5e^{-t}, & 0 \le t < \pi \\ 5e^{-t}+\frac{3}{2}\left(e^{-(t-\pi)}-\cos(t)-\sin(t)\right) & \pi \le t \end{cases}$$

BEAMS

$EI \frac{d^4y}{dx^4} = w(x)$. $y(x)$ is deflection of beam of length L, carrying a load of $w(x)$ per unit length.

E is Young's modulus of elasticity and

I is a moment of inertia of a cross-section of the beam

This 7.3 method is good for these kinds of eqns, when $w(x)$ is piecewise-defined.

Example 11 A Boundary-Value Problem

A beam of length L is embedded at both ends, as shown in Figure 7.3.8. Find the deflection of the beam when the load is given by

$$w(x) = \begin{cases} w_0\left(1 - \dfrac{2}{L}x\right), & 0 < x < L/2 \\ 0, & L/2 < x < L. \end{cases}$$

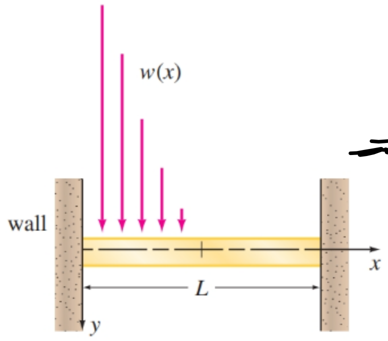


Figure 7.3.8 Embedded beam with variable load in Example 11

$$EI\frac{d^4y}{dx^4} = w(x) = w_0\left(1-\frac{2}{L}x\right) - \mathcal{U}\left(x-\frac{L}{2}\right)w_0\left(1-\frac{2}{L}x\right)$$

$$\Rightarrow EI\mathcal{L}(y^{(4)}) = \mathcal{L}\left(w_0\left(1-\frac{2}{L}x\right)\right) - \mathcal{L}\left(\mathcal{U}\left(x-\frac{L}{2}\right)w_0\left(1-\frac{2}{L}x\right)\right)$$

$$EI\left[s^4\mathcal{L}(y) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\right]$$

$$= w_0\mathcal{L}(1) - \frac{2w_0}{L}\mathcal{L}(x) - \mathcal{L}\left(\mathcal{U}\left(x-\frac{L}{2}\right)w_0\left(\frac{L}{2}-x\right)\left(\frac{2}{L}\right)\right)$$

$$= w_0\left(\frac{1}{s}\right) - \frac{2w_0}{L}\left(\frac{1}{s^2}\right) + \mathcal{L}\left(\mathcal{U}\left(x-\frac{L}{2}\right)w_0\left(\frac{2}{L}\right)\left(x-\frac{L}{2}\right)\right)$$

$$= w_0\left(\frac{1}{s}\right) - \frac{2w_0}{L}\left(\frac{1}{s^2}\right) + \frac{2w_0}{L}\left(\mathcal{L}\left(\mathcal{U}\left(x-\frac{L}{2}\right)\left(x-\frac{L}{2}\right)\right)\right)$$

$$= w_0\left(\frac{1}{s}\right) - \frac{2w_0}{L}\left(\frac{1}{s^2}\right) + \frac{2w_0}{L}\left[e^{-\frac{L}{2}s}\left(\frac{1}{s^2}\right)\right]$$

By the physical situation, $y(0) = y'(0) = y(L) = y'(L) = 0$

why doesn't that mean that $y'''(0) = y^{(4)}(0) = 0$?

$$EI\left[s^4\mathcal{L}(y) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\right]$$

$$= EI\left[s^4\mathcal{L}(y) - sy''(0) - y'''(0)\right]$$

$$= EI\left[s^4\mathcal{L}(y) - c_1s - c_2\right], \text{ where } c_1 \equiv y''(0),\ c_2 \equiv y'''(0)$$

$$\Rightarrow EI\left[s^4\mathcal{L}(y) - c_1s - c_2\right] = w_0\left(\frac{1}{s}\right) - \frac{2w_0}{L}\left(\frac{1}{s^2}\right) + \frac{2w_0}{L}\left[e^{-\frac{L}{2}s}\left(\frac{1}{s^2}\right)\right]$$

$$\Rightarrow s^4\mathcal{L}(y) - c_1s - c_2 = \frac{1}{EI}\left[w_0\left(\frac{1}{s}\right) - \frac{2w_0}{L}\left(\frac{1}{s^2}\right) + \frac{2w_0}{L}\left[e^{-\frac{L}{2}s}\left(\frac{1}{s^2}\right)\right]\right]$$

$$\Rightarrow s^4\mathcal{L}(y) = \frac{1}{EI}\left[w_0\left(\frac{1}{s}\right) - \frac{2w_0}{L}\left(\frac{1}{s^2}\right) + \frac{2w_0}{L}\left[e^{-\frac{L}{2}s}\left(\frac{1}{s^2}\right)\right]\right] + c_1s + c_2$$

$$\Rightarrow \mathcal{L}(y) = \frac{1}{EI}\left[w_0\left(\frac{1}{s^5}\right) - \frac{2w_0}{L}\left(\frac{1}{s^6}\right) + \frac{2w_0}{L}e^{-\frac{L}{2}s}\left(\frac{1}{s^6}\right)\right] + \frac{c_1}{s^3} + \frac{c_2}{s^4}$$

$$= \frac{1}{EI}\left[w_0 \cdot \frac{1}{4!}\left(\frac{4!}{s^5}\right) - \frac{2w_0}{5!L}\left(\frac{5!}{s^6}\right) + \frac{2w_0}{5!L}e^{-\frac{L}{2}s}\left(\frac{5!}{s^6}\right)\right] + \frac{c_1}{2!}\cdot\frac{2!}{s^3} + \frac{c_2}{3!}\cdot\frac{3!}{s^4}$$

$$\Rightarrow y = \frac{w_0}{4!EI}\left(x^4\right) - \left(\frac{2w_0}{5!LEI}\right)\left(x^5\right) + \frac{2w_0}{EI\cdot 5!L}\left(x-\frac{L}{2}\right)^5\mathcal{U}\left(x-\frac{L}{2}\right) + \frac{c_1}{2}x^2 + \frac{c_2}{6}x^3$$

$$= \frac{c_1}{2}x^2 + \frac{c_2}{6}x^3$$

My coefficients are slightly off, but I think I got the general idea, here.