Section 7.2 - Inverse Transforms and Transforms of Derivatives

We're almost there. The idea is to take the Laplace Transform of a *differential equation*, solve the resulting system of linear equations (essentially), followed by taking the *inverse* Laplace Transform to obtain the solution *y*.

So bear with these formalities.

Inverse Functions are nothing new. We're just expanding our concept of functions or operators, especially linear operators.

$x+3=0$

we're inverting the operator $L:\mathbb{R}\to\mathbb{R}$, given by $L(x)=x+3$

$L^{-1}(x)=x-3$. We subtract the 3 to "invert" the addition operator. So it goes all the way back to Intermediate Algebra.

$$\ln(x)=7 \Rightarrow L(x)=\ln(x) \Rightarrow L^{-1}(x)=e^x \Rightarrow$$

$$L(x)=7 \Rightarrow$$

$$L^{-1}(L(x))=L^{-1}(7)=\boxed{e^7=x}$$

Optional

The classic linear transform is multiplication on the left by a matrix. Let A be a 2×2 matrix and $\bar{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ be a 2×1 matrix with variable entries. Let $\bar{b}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$

$A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$.

$A:\mathbb{R}^2\to\mathbb{R}^2$ given by $A(\bar{x})=A\bar{x}$ is a linear operator.

$A\begin{bmatrix}0\\0\end{bmatrix}=\bar{0}$ ✓

$A(c\bar{b}+c\bar{d})=cA\bar{b}+cA\bar{d}$

Consider the system:

$$(1)\quad \begin{aligned} a_{11}x_1+a_{12}x_2&=b_1\\ a_{21}x_1+a_{22}x_2&=b_2\end{aligned}$$

Consider $A\bar{x}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}a_{11}x_1+a_{12}x_2\\ a_{21}x_1+a_{22}x_2\end{bmatrix}$

holds the LHS of (1).

It's very hard to teach this class, while withholding this.

And

$A\bar{x}=\bar{b}$ is an elegant way of expressing (1)

$\bar{x}=A^{-1}(Ax)=A^{-1}\bar{b}$ is the solution.

Now, think of $\mathcal{L}$ that way

$$a_2y''+a_1y'+a_0y=f(t)$$

$$\mathcal{L}(a_2y'' + a_1y' + a_0y) = \mathcal{L}(f(t)) = F(s)$$

$$a_2P_2(s)Y(s) + a_1P_1(s)Y(s) + a_0P_0(s)Y(s) = F(s)$$

$$Y(s)\Big[\underbrace{a_2P_2(s) + a_1P_1(s) + a_0P(s)}_{Q(s)}\Big] = F(s)$$

$$Y(s)Q(s) = F(s)$$

$$Y(s) = \frac{F(s)}{Q(s)}$$

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}(\mathcal{L}(y)) = \mathcal{L}^{-1}\left(\frac{F(s)}{Q(s)}\right)$$

$$y = \mathcal{L}^{-1}\left(\frac{F(s)}{Q(s)}\right)$$

$\mathcal{L}^{-1}$ is the analogue of A^{-1}, above.

It turns out that evaluating $\mathcal{L}^{-1}$ involves integrating by partial fractions, and partial fractions ALWAYS involves a system of simultaneous equations that can be written as a matrix equation

$A\bar{x} = \bar{b}$ and

A is always going to have an inverse when

the coefficients of $Ly = a_2y'' + a_1y' + a_0y$, a_0, a_1, a_2, are real (or complex) constants

We're going to keep the order down to 2nd-order, Here's what we need

$$\mathcal{L}y = Y(s)$$
$$\mathcal{L}y' = sY(s) - y(0)$$
$$\mathcal{L}y'' = s^2 Y(s) - s y(0) - y'(0)$$

} This is as far as we'll go.

$$\vdots$$
$$\mathcal{L}y^{(n)} = s^n Y(s) - s^{n-1} y(0) - s^{(n-2)} y'(0) - s^{(n-3)} y^{(2)}(0) - \cdots - y^{(n-1)}(0)$$

Now, if the a_k's are constant, $a_2 y'' + a_1 y' + a_0 y = f(t)$

$$a_2\left(s^2 Y(s) - s y(0) - y'(0)\right) + a_1\left(sY(s) - y(0)\right) + a_0 Y(s) = F(s)$$

$$\Rightarrow \left(a_2 s^2 + a_1 s + a_0\right) Y(s) - a_2 s y(0) - a_2 y'(0) - a_1 y(0) = F(s)$$

$\Rightarrow P(s)Y(s) - Q(s) = F(s)$, where

$P(s) = a_2 s^2 + a_1 s + a_0 \quad \left(= \sum_{k=0}^{n} a_{n-k} s^{n-k}\right)$ and

$$Q(s) = a_2 s y(0) + a_2 y'(0) + a_1 y(0)$$

$$Q_3(s) = a_3 s^2 y(0) + a_3 s y'(0) + a_3 y''(0) + a_2 s y(0) + a_2 y'(0) + a_1 y(0)$$

$= \sum_{k=0}^{3} \sum_{j=0}^{n-k} a_{3-j} s^{3-j-1} y^{(j)}(0)$ is taking a stab at it.

$= \sum_{k=0}^{n}\left(\sum_{j=0}^{n-k} a_{n-j} s^{n-j-1} y^{(j)}(0)\right)$, I think.

If you can write it, you can program it.

Mastering this "high-level" language is the key to translating these symbols into computer-algebra syntax.

↳ Also very high-level
Down from that might be
LISP, FORTRAN, Assembler, Binary.

Example 4

$Ly = y' + 3y = 13\sin(2t),\ y(0) = 6$

$\mathcal{L}(Ly) = sY(s) - y(0) + 3Y(s) = 13\mathcal{L}(\sin(2t))$

$= sY(s) - 6 + 3Y(s) = 13\left(\frac{2}{s^2+2^2}\right)$

$(s+3)Y(s) = \frac{26}{s^2+2^2} + 6 = \frac{26 + 6s^2 + 24}{s^2+2^2} \longrightarrow$

$Y(s) = \frac{6s^2+50}{(s+3)(s^2+4)}$

$\mathcal{L}^{-1}(Y(s)) = y(t) = \mathcal{L}^{-1}\left(\frac{6s^2+50}{(s+3)(s^2+4)}\right)$

$\frac{6s^2+50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$

$6s^2+50 = As^2 + 4A + (Bs+C)(s+3)$

$s = -3 \longrightarrow$

$6(9)+50 = 9A + 4A + (-3B+C)(0)$

$54+50 = 104 = 13A$

$A = \frac{104}{13} = 8$

$6s^2+50 = 8(s^2+4) + (Bs+C)(s+3)$

$= 8s^2 + 32 + Bs^2 + 3Bs + Cs + 3C \longrightarrow$

$-2s^2 = Bs^2$

$\Rightarrow \boxed{B = -2}$

$\cancel{6s^2} + 50 = \cancel{8s^2} + 32 - \cancel{2s^2} - 6Bs + Cs + 3C$

$-6Bs + Cs = 0$

$3C + 32 = 50$

$3C = 18$

$\boxed{C = 6}$

$Y(s) = \frac{6s^2+50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4} = \frac{8}{s+3} - \frac{2s}{s^2+4} + \frac{6}{s^2+4}$

$\Rightarrow y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{8}{s+3} - \frac{2s}{s^2+4} + \frac{6}{s^2+4}\right) = 8e^{-3t} - 2\cos(2t)$

$+ \mathcal{L}^{-1}\left(3 \cdot \frac{2}{s^2+2^2}\right)$

$= \boxed{8e^{-3t} - 2\cos(2t) + 3\sin(2t) = y(t)}$

using Theorem 7.2.1