7.1 - Definition of the Laplace Transform

Recall, linearity property of $\frac{d*}{dx}$, $\int * dx$

$a, b \in \mathbb{R}$, f, g diff'ble functions $\longrightarrow$

$$\frac{d}{dx}\left[af(x)+bg(x)\right] = a\frac{df}{dx} + b\frac{dg}{dx}$$

$a, b \in \mathbb{R}$, f, g cnt's functions $\longrightarrow$

$$\int (af(x)+bg(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

Same for definite integral

$$\int_{x_0}^{x_1} (af(x)+bg(x))\,dx = a\int_{x_0}^{x_1} f(x)\,dx + b\int_{x_0}^{x_1} g(x)\,dx$$

Note that $\int_{x_0}^{x_1} K(s,t)\,dt$ is a function of s, if $K(s,t)$ is cnt's in s & t.

Recall: Improper Integral

$$\int_0^\infty K(s,t)\,dt = \lim_{b\to\infty}\int_0^b K(s,t)\,dt$$

If the limit exists, then $\int_0^\infty K(s,t)\,dt$ converges.

Ⓦ $\int_0^\infty K(s,t)\,dt$ diverges.

If $f(t)$ defined for $t \geq 0 \longrightarrow$

$$F(s) = \int_0^\infty K(s,t)f(t)\,dt$$ "transforms" $f(t)$ into $F(s)$.

<u>Laplace Transform</u>: $\mathcal{L}(f(t)) = \int_0^\infty e^{-st}f(t)\,dt$.

$\mathcal{D}(F) = $ Domain of $F = \{s \mid \mathcal{L}(f)$ converges.

For us, we assume $s \in \mathbb{R}$ ($s \in \mathbb{C}$ more generally)

We dispense with "lim" & just write

$\lim_{b\to\infty} F(s)\Big|_0^b = F(s)\Big|_0^\infty$ with the understanding that it's still a limit.

$\boxed{E2}$ $\mathcal{L}(t) = \int_0^\infty e^{-st} \cdot t \, dt$

$$\int_0^{\text{infinity}} \exp(-s \cdot t) \cdot t \, dt = \lim_{t \to \infty} \left(-\frac{e^{-st} s t + e^{-st} - 1}{s^2} \right)$$

$$= \lim_{t \to \infty} \left[\frac{-ste^{-st}}{s^2} - \frac{e^{-st}}{s^2} + \frac{1}{s^2} \right] = \boxed{\frac{1}{s^2} = \mathcal{L}(t).}$$

Maple doesn't like this. But if we tell it that $s > 0$ ahead of time:

$assume(s > 0)$

$$\int_0^{\text{infinity}} \exp(-s \cdot t) \cdot t \, dt = \frac{1}{s\sim^2}$$

So you get the idea. But we're not going to be evaluating a ton of these. A few, here in 7.1, but mostly we'll refer to the following table (in the back of the textbook):

	$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1.	1	$\dfrac{1}{s}$
2.	t	$\dfrac{1}{s^2}$
3.	t^n	$\dfrac{n!}{s^{n+1}}, \; n \in \mathbb{N}$
4.	$t^{-1/2}$	$\dfrac{\sqrt{\pi}}{s^{1/2}}$
5.	$t^{1/2}$	$\dfrac{\sqrt{\pi}}{2s^{3/2}}$
6.	t^{α}	$\dfrac{\Gamma(\alpha + 1)}{s^{\alpha+1}}, \quad \alpha > -1$
7.	$\sin kt$	$\dfrac{k}{s^2 + k^2}$
8.	$\cos kt$	$\dfrac{s}{s^2 + k^2}$
9.	$\sin^2 kt$	$\dfrac{2k^2}{s(s^2 + 4k^2)}$
10.	$\cos^2 kt$	$\dfrac{s^2 + 2k^2}{s(s^2 + 4k^2)}$
11.	e^{at}	$\dfrac{1}{s - a}$
12.	$\sinh kt$	$\dfrac{k}{s^2 - k^2}$
13.	$\cosh kt$	$\dfrac{s}{s^2 - k^2}$
14.	$\sinh^2 kt$	$\dfrac{2k^2}{s(s^2 - 4k^2)}$
15.	$\cosh^2 kt$	$\dfrac{s^2 - 2k^2}{s(s^2 - 4k^2)}$
16.	$e^{at}\,t$	$\dfrac{1}{(s - a)^2}$
17.	$e^{at}\,t^n$	$\dfrac{n!}{(s - a)^{n+1}},$
18.	$e^{at} \sin kt$	$\dfrac{k}{(s - a)^2 + k^2}$
19.	$e^{at} \cos kt$	$\dfrac{s - a}{(s - a)^2 + k^2}$
20.	$e^{at} \sinh kt$	$\dfrac{k}{(s - a)^2 - k^2}$

I'll do one more:

$\boxed{E3}$ (a) $\mathcal{L}(e^{-3t})dt$

$$\int_0^\infty e^{-st}e^{-3t}dt = \int_0^\infty e^{-(s+3)t}dt = -\frac{1}{s+3}e^{-(s+3)t}\Big|_0^\infty =$$

$$= -\frac{1}{s+3}\left[0 - (e^{-0})\right] = \frac{1}{s+3} = \mathcal{L}(e^{-3t}) = F(s)$$

Note: $\mathcal{D}(F) = \{s \mid s > -3\}$

Sufficient conditions for existence of $\mathcal{L}(f(t))$:

f must be piecewise-cont's and be of EXPONENTIAL ORDER:

Exponential order:

Not piecewise-continuous.
No ∞ discontinuities.

Definition 7.1.2 Exponential Order

A function f defined on $[0, \infty)$ is said to be of **exponential order** as $t \to \infty$ if there exist constants c, $M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$.

Basically grow less fast than an exponential.

$$|f(t)| \leq Me^{ct} \quad \forall t > T$$

iff

$$\left|\frac{f(t)}{e^{ct}}\right| \leq M \quad \forall t > T$$

iff

$$\left|\frac{f(t)}{e^{ct}}\right| \xrightarrow{t \to \infty} L \leq M$$

(E) $f(t) = t^5$:

$$\left|\frac{t^5}{e^{ct}}\right| \xrightarrow[L'H]{t \to \infty} \frac{5t^4}{ce^{ct}} \xrightarrow[L'H]{t \to \infty} \frac{5 \cdot 4t^3}{c^2 e^{ct}} \xrightarrow[L'H]{t \to \infty} \frac{5 \cdot 4 \cdot 3t^2}{c^3 e^{ct}}$$

$$\xrightarrow[L'H]{t \to \infty} \frac{5 \cdot 4 \cdot 3 \cdot 2t}{c^4 e^{ct}} \xrightarrow[L'H]{t \to \infty} \frac{5!}{c^5 e^{ct}} \xrightarrow{t \to \infty} 0 < M \; \forall M > 0$$

Theorem 7.1.2 Sufficient Conditions for Existence

If a function f is piecewise continuous on $[0, \infty)$ and of exponential order with c as specified in Definition 7.1.2, then $\mathscr{L}\{f(t)\}$ exists for $s > c$.

Sufficiency is "enough or more than enough." You might be able to weaken the condition.

$$\mathscr{L}\{f(t)\} = \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt = I_1 + I_2.$$

piecewise-

$e^{-st} f(t)$ cnts. $[0,T]$ bdd $\Rightarrow I_1$ is finite

$I_2 = \int_T^\infty e^{-st} f(t)\,dt \le \int_T^\infty e^{-st}(Me^{ct})\,dt = M\int_T^\infty e^{-(s-c)t}\,dt$ converges

$\forall s > c$. $\blacksquare$

Stated without proof. Ask me if you want the proof. It's fairly straightforward, but bears some explanation.

Theorem 7.1.3 Behavior of $F(s)$ as $s \to \infty$

If a function f is piecewise continuous on $[0, \infty)$ and of exponential order with c as specified in Definition 7.1.2 and $\mathscr{L}\{f(t)\} = F(s)$, then $\lim_{s\to\infty} F(s) = 0$.

In Problems 1–18 use Definition 7.1.1 to find $\mathscr{L}\{f(t)\}$.

5. $f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

Book Answer:

Exercises 7.1 (Page 292)

1. $\frac{2}{s}e^{-s} - \frac{1}{s}$

5. $\frac{1 + e^{-\pi s}}{s^2 + 1}$

Need: **assume($s > 0$)**

Maple Answer:

$$\int_0^{\text{infinity}} e^{-s \cdot t} \cdot f(t)\, dt = \frac{1 + e^{-\pi s\sim}}{s\sim^2 + 1}$$

The "s~" is from the "assume(s>0)" command.

Evaluating these is a lot of tedious integration by parts. Maple speeds things up quite a bit.

New Command: piecewise

$f := t \to piecewise(0 \le t < \pi,\ sin(t),\ t \ge \pi,\ 0)$

$$f := t \mapsto \begin{cases} \sin(t) & 0 \le t < \pi \\ 0 & \pi \le t \end{cases}$$

I was asked about #1 in 7.1 in class, today. I think the exponential definitions of sinh(*t*) and cosh(*t*) were all that was needed, in order to evaluate the improper integral:

$\sinh t = \frac{e^{t} - e^{-t}}{2}$

$\cosh t = \frac{e^{t} + e^{-t}}{2}$

$e^{-st}\left(\frac{e^{t} + e^{-t}}{2}\right)$

$= \frac{1}{2} e^{-st+t} + \frac{1}{2} e^{-st-t}$

$= \frac{1}{2} e^{-(s-1)t} + \frac{1}{2} e^{-(s+1)t}$

$\mathcal{L}(\cosh(t)) = \frac{1}{2}\int_0^\infty e^{-(s-1)t}\,dt + \frac{1}{2}\int_0^\infty e^{-(s+1)t}\,dt$

$-\frac{1}{s-1} \cdot \frac{1}{2} e^{-(s-1)t} \Big|_0^\infty + \ldots$