Section 6.3 - Power Series about Singular Points

We derive the general form of the Indicial Equation

$unassign('y(x)')$

$p := x \to \sum_{n=0}^{2} (a[n] \cdot x^n)$

$$p := x \mapsto \sum_{n=0}^{2} a_n \cdot x^n \quad \textbf{(1.1)}$$

$p(x)$

$$x^2 a_2 + a_1 x + a_0 \quad \textbf{(1.2)}$$

$sort(p(x), x, ascending)$

$$a_0 + a_1 x + a_2 x^2 \quad \textbf{(1.3)}$$

$q := x \to \sum_{n=0}^{2} (b[n] \cdot x^n)$

$$q := x \mapsto \sum_{n=0}^{2} b_n \cdot x^n \quad \textbf{(1.4)}$$

$q(x)$

$$x^2 b_2 + x b_1 + b_0 \quad \textbf{(1.5)}$$

$sort(q(x), x, ascending)$

$$b_0 + b_1 x + b_2 x^2 \quad \textbf{(1.6)}$$

$y := x \to \sum_{n=0}^{2} (c[n] \cdot x^{n+r})$

$$y := x \mapsto \sum_{n=0}^{2} c_n \cdot x^{n+r} \quad \textbf{(1.7)}$$

$eqn := x^2 \cdot diff(diff(y(x), x), x) + x \cdot p(x) \cdot diff(y(x), x) + q(x) \cdot y(x)$

$$eqn := x^2 \left(\frac{c_0 x^r r^2}{x^2} - \frac{c_0 x^r r}{x^2} + \frac{c_1 x^{1+r} (1+r)^2}{x^2} - \frac{c_1 x^{1+r} (1+r)}{x^2} + \frac{c_2 x^{2+r} (2+r)^2}{x^2} - \frac{c_2 x^{2+r} (2+r)}{x^2} \right) + x \left(a_0 + a_1 x + a_2 x^2\right) \left(\frac{c_0 x^r r}{x} + \frac{c_1 x^{1+r} (1+r)}{x} + \frac{c_2 x^{2+r} (2+r)}{x} \right) + \left(b_0 + b_1 x + b_2 x^2\right) \left(c_0 x^r + c_1 x^{1+r} + c_2 x^{2+r}\right) \quad \textbf{(1.8)}$$

$expand(\%)$

$$x^r r^2 c_0 + c_0 x^r r a_2 x^2 + c_0 x^r r a_1 x + c_0 x^r r a_0 + c_1 x^r r a_2 x^3 + c_1 x^r r a_1 x^2 + c_1 x^r r a_0 x \quad \textbf{(1.9)}$$

$$+ x^r r c_2 a_2 x^4 + x^r r c_2 a_1 x^3 + x^r r c_2 a_0 x^2 + c_0 x^r x^2 b_2 + c_0 x^r x b_1 + c_1 x^r x^3 b_2 + c_1 x^r x^2 b_1 + c_1 x^r x b_0 + c_2 x^r x^4 b_2 + c_2 x^r x^3 b_1 + c_2 x^r x^2 b_0 + c_1 x^r a_1 x^2 + c_1 x^r a_0 x + 2 x^r c_2 a_2 x^4 + 2 x^r c_2 a_1 x^3 + 2 x^r c_2 a_0 x^2 - c_0 x^r r + c_1 x^r r^2 x + c_1 x^r r x + x^r r^2 c_2 x^2 + 3 c_2 x^r r x^2 + c_1 x^r a_2 x^3 + 2 c_2 x^r x^2 + x^r b_0 c_0$$

collect(%, *x*)

$$x^r r^2 c_0 + c_0 x^r r a_0 - c_0 x^r r + x^r b_0 c_0 + \left(x^r r^2 c_1 + c_1 x^r r a_0 + c_0 x^r r a_1 + c_1 x^r r + c_1 x^r a_0 + x^r b_0 c_1 + x^r b_1 c_0\right) x + \left(x^r r^2 c_2 + x^r r a_0 c_2 + x^r r a_1 c_1 + x^r r a_2 c_0 + 3 c_2 x^r r + 2 x^r a_0 c_2 + x^r a_1 c_1 + x^r b_0 c_2 + x^r b_1 c_1 + x^r b_2 c_0 + 2 c_2 x^r\right) x^2 + \left(x^r r a_1 c_2 + x^r r a_2 c_1 + 2 x^r a_1 c_2 + x^r a_2 c_1 + x^r b_1 c_2 + x^r b_2 c_1\right) x^3 + \left(x^r r a_2 c_2 + 2 x^r a_2 c_2 + x^r b_2 c_2\right) x^4 \quad \textbf{(1.10)}$$

sort(%, *x*, *ascending*)

$$x^r r^2 c_0 + c_0 x^r r a_0 - c_0 x^r r + x^r b_0 c_0 + \left(x^r r^2 c_1 + c_1 x^r r a_0 + c_0 x^r r a_1 + c_1 x^r r + c_1 x^r a_0 + x^r b_0 c_1 + x^r b_1 c_0\right) x + \left(x^r r^2 c_2 + x^r r a_0 c_2 + x^r r a_1 c_1 + x^r r a_2 c_0 + 3 c_2 x^r r + 2 x^r a_0 c_2 + x^r a_1 c_1 + x^r b_0 c_2 + x^r b_1 c_1 + x^r b_2 c_0 + 2 c_2 x^r\right) x^2 + \left(x^r r a_1 c_2 + x^r r a_2 c_1 + 2 x^r a_1 c_2 + x^r a_2 c_1 + x^r b_1 c_2 + x^r b_2 c_1\right) x^3 + \left(x^r r a_2 c_2 + 2 x^r a_2 c_2 + x^r b_2 c_2\right) x^4 \quad \textbf{(1.11)}$$

$zerothterm := x^r r^2 c_0 + c_0 x^r r a_0 - c_0 x^r r + x^r b_0 c_0$

$$zerothterm := x^r r^2 c_0 + c_0 x^r r a_0 - c_0 x^r r + x^r b_0 c_0 \quad \textbf{(1.12)}$$

$simplify\left(\frac{zerothterm}{x^r \cdot c[0]}\right)$

$$r^2 + \left(a_0 - 1\right) r + b_0 \quad \textbf{(1.13)}$$

expand(%)

$$r^2 + r a_0 - r + b_0 \quad \textbf{(1.14)}$$

This confirms the book formula for the General Indicial Equation: $r(r-1) + ra_0 + b_0 = 0$.

#8

Explore series command.

?*series*

$series(\sin(x), x = 0, 8)$

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + O(x^9) \tag{2.1.1}$$

unassign('y(x)')

eqn := x·*diff*(*diff*(y(x), x), x) − x·*diff*(y(x), x) + y(x) = 0

$$eqn := x\left(\frac{d^2}{dx^2}\,y(x)\right) - x\left(\frac{d}{dx}\,y(x)\right) + y(x) = 0 \tag{2.1}$$

I'm using $n = 5$, but you should also play with n = infinity.

$y := x \to \sum_{n=0}^{5} \left(c[n]\cdot x^{n+r}\right)$

$$y := x \mapsto \sum_{n=0}^{5} c_n \cdot x^{n+r} \tag{2.2}$$

eqn

$$x\left(\frac{x^r r^2 c_0}{x^2} - \frac{x^r r c_0}{x^2} + \frac{c_1 x^{r+1}(r+1)^2}{x^2} - \frac{c_1 x^{r+1}(r+1)}{x^2} + \frac{x^{r+2}(r+2)^2 c_2}{x^2} - \frac{x^{r+2}(r+2)c_2}{x^2} + \frac{x^{r+3}(r+3)^2 c_3}{x^2} - \frac{x^{r+3}(r+3)c_3}{x^2} + \frac{x^{r+4}(r+4)^2 c_4}{x^2} - \frac{x^{r+4}(r+4)c_4}{x^2} + \frac{c_5 x^{5+r}(5+r)^2}{x^2} - \frac{c_5 x^{5+r}(5+r)}{x^2}\right) - x\left(\frac{x^r r c_0}{x} + \frac{c_1 x^{r+1}(r+1)}{x} + \frac{x^{r+2}(r+2)c_2}{x} + \frac{x^{r+3}(r+3)c_3}{x} + \frac{x^{r+4}(r+4)c_4}{x} + \frac{c_5 x^{5+r}(5+r)}{x}\right) + x^r c_0 + c_1 x^{r+1} + x^{r+2}c_2 + x^{r+3}c_3 + x^{r+4}c_4 + c_5 x^{5+r} = 0 \tag{2.3}$$

expand(%)

$$2x^r c_2 x + 6x^r c_3 x^2 + 12x^r c_4 x^3 - 3x^r c_4 x^4 - 2x^r c_3 x^3 - x^r c_2 x^2 - x^r r c_4 x^4 - x^r r c_3 x^3 - x^r r c_2 x^2 - x^r r c_1 x + x^r r^2 c_4 x^3 + x^r r^2 c_3 x^2 + x^r r^2 c_2 x + x^r r^2 c_1 + \frac{x^r r^2 c_0}{x} + 7x^r r c_4 x^3 + 5x^r r c_3 x^2 + 3x^r r c_2 x + x^r r c_1 - \frac{x^r r c_0}{x} - 4c_5 x^5 x^r + 20c_5 x^4 x^r - c_0 x^r r + c_5 x^4 x^r r^2 + 9c_5 x^4 x^r r - c_5 x^5 x^r r + x^r c_0 = 0 \tag{2.4}$$

collect(%, x)

$$x^r r^2 c_1 + x^r r c_1 - c_0 x^r r + x^r c_0 + \frac{x^r r^2 c_0 - c_0 x^r r}{x} + \left(x^r r^2 c_2 - x^r r c_1 + 3x^r r c_2 + 2x^r c_2\right)x + \left(x^r r^2 c_3 - x^r r c_2 + 5x^r r c_3 - x^r c_2 + 6x^r c_3\right)x^2 + \left(x^r r^2 c_4 - x^r r c_3 + 7x^r r c_4 - 2x^r c_3 + 12x^r c_4\right)x^3 + \left(c_5 x^r r^2 - x^r r c_4 + 9c_5 x^r r - 3x^r c_4 + 20c_5 x^r\right)x^4 + \left(-c_5 x^r r - 4c_5 x^r\right)x^5 \tag{2.5}$$

$$= 0$$

$sort(\%, x, ascending)$

$$x^r r^2 c_1 + x^r r c_1 - c_0 x^r r + x^r c_0 + \frac{x^r r^2 c_0 - c_0 x^r r}{x} + \left(x^r r^2 c_2 - x^r r c_1 + 3 x^r r c_2 + 2 x^r c_2\right) x + \left(x^r r^2 c_3 - x^r r c_2 + 5 x^r r c_3 - x^r c_2 + 6 x^r c_3\right) x^2 + \left(x^r r^2 c_4 - x^r r c_3 + 7 x^r r c_4 - 2 x^r c_3 + 12 x^r c_4\right) x^3 + \left(c_5 x^r r^2 - x^r r c_4 + 9 c_5 x^r r - 3 x^r c_4 + 20 c_5 x^r\right) x^4 + \left(-c_5 x^r r - 4 c_5 x^r\right) x^5 = 0 \quad \textbf{(2.6)}$$

Let's clear out the x^r:

$\frac{\%}{x^r}$

$$\frac{1}{x^r}\left(x^r r^2 c_1 + x^r r c_1 - c_0 x^r r + x^r c_0 + \frac{x^r r^2 c_0 - c_0 x^r r}{x} + \left(x^r r^2 c_2 - x^r r c_1 + 3 x^r r c_2 + 2 x^r c_2\right) x + \left(x^r r^2 c_3 - x^r r c_2 + 5 x^r r c_3 - x^r c_2 + 6 x^r c_3\right) x^2 + \left(x^r r^2 c_4 - x^r r c_3 + 7 x^r r c_4 - 2 x^r c_3 + 12 x^r c_4\right) x^3 + \left(c_5 x^r r^2 - x^r r c_4 + 9 c_5 x^r r - 3 x^r c_4 + 20 c_5 x^r\right) x^4 + \left(-c_5 x^r r - 4 c_5 x^r\right) x^5\right) = 0 \quad \textbf{(2.7)}$$

$expand(\%)$

$$-c_0 r + c_0 + r^2 c_1 + r c_1 + \frac{r^2 c_0}{x} - \frac{r c_0}{x} - r c_1 x + r^2 c_2 x + 3 r c_2 x + 2 c_2 x - r c_2 x^2 + r^2 c_3 x^2 + 5 r c_3 x^2 + 6 c_3 x^2 - c_2 x^2 - r c_3 x^3 + r^2 c_4 x^3 + 7 r c_4 x^3 + 12 c_4 x^3 - 2 c_3 x^3 + c_5 r^2 x^4 + 9 c_5 r x^4 + 20 c_5 x^4 - r c_4 x^4 - 3 c_4 x^4 - c_5 r x^5 - 4 c_5 x^5 = 0 \quad \textbf{(2.8)}$$

$sort(\%, x, ascending)$

$$-c_0 r + c_0 + r^2 c_1 + r c_1 + \frac{r^2 c_0}{x} - \frac{r c_0}{x} - r c_1 x + r^2 c_2 x + 3 r c_2 x + 2 c_2 x - r c_2 x^2 + r^2 c_3 x^2 + 5 r c_3 x^2 + 6 c_3 x^2 - c_2 x^2 - r c_3 x^3 + r^2 c_4 x^3 + 7 r c_4 x^3 + 12 c_4 x^3 - 2 c_3 x^3 + c_5 r^2 x^4 + 9 c_5 r x^4 + 20 c_5 x^4 - r c_4 x^4 - 3 c_4 x^4 - c_5 r x^5 - 4 c_5 x^5 = 0 \quad \textbf{(2.9)}$$

$collect(\%, x)$

$$r^2 c_1 - c_0 r + r c_1 + c_0 + \frac{r^2 c_0 - c_0 r}{x} + \left(r^2 c_2 - r c_1 + 3 r c_2 + 2 c_2\right) x + \left(r^2 c_3 - r c_2 + 5 r c_3 - c_2 + 6 c_3\right) x^2 + \left(r^2 c_4 - r c_3 + 7 r c_4 - 2 c_3 + 12 c_4\right) x^3 + \left(r^2 c_5 - r c_4 + 9 r c_5 - 3 c_4 + 20 c_5\right) x^4 + \left(-r c_5 - 4 c_5\right) x^5 = 0 \quad \textbf{(2.10)}$$

We can see the Indicial Equation peeking at us from the $\frac{1}{x}$ term.

$sort(\%, x, ascending)$

$$r^2 c_1 - c_0 r + r c_1 + c_0 + \frac{r^2 c_0 - c_0 r}{x} + \left(r^2 c_2 - r c_1 + 3 r c_2 + 2 c_2\right) x + \left(r^2 c_3 - r c_2 + 5 r c_3 - c_2 + 6 c_3\right) x^2 + \left(r^2 c_4 - r c_3 + 7 r c_4 - 2 c_3 + 12 c_4\right) x^3 + \left(r^2 c_5 - r c_4 + 9 r c_5 - 3 c_4 + 20 c_5\right) x^4 + \left(-r c_5 - 4 c_5\right) x^5 = 0 \quad \textbf{(2.11)}$$

$subs(r = 1, \%)$

$$2 c_1 + \left(6 c_2 - c_1\right) x + \left(12 c_3 - 2 c_2\right) x^2 + \left(20 c_4 - 3 c_3\right) x^3 + \left(30 c_5 - 4 c_4\right) x^4 - 5 c_5 x^5 = 0 \quad \textbf{(2.12)}$$

This $r = 1$ thing isn't very conducive. Copy-paste from before we subbed in $r = 1$ and try $r = 0$.

What I'm doing is a bit clumsier than what you might do, because I wanted you to see the false path.

$$r^2 c_1 - c_0 r + r c_1 + c_0 + \frac{r^2 c_0 - c_0 r}{x} + \left(r^2 c_2 - r c_1 + 3 r c_2 + 2 c_2\right) x + \left(r^2 c_3 - r c_2 + 5 r c_3 - c_2 + 6 c_3\right) x^2 + \left(r^2 c_4 - r c_3 + 7 r c_4 - 2 c_3 + 12 c_4\right) x^3 + \left(r^2 c_5 - r c_4 + 9 r c_5 - 3 c_4 + 20 c_5\right) x^4 + \left(-r c_5 - 4 c_5\right) x^5 = 0$$

$$r^2 c_1 - c_0 r + r c_1 + c_0 + \frac{r^2 c_0 - c_0 r}{x} + \left(r^2 c_2 - r c_1 + 3 r c_2 + 2 c_2\right) x + \left(r^2 c_3 - r c_2 + 5 r c_3 - c_2 + 6 c_3\right) x^2 + \left(r^2 c_4 - r c_3 + 7 r c_4 - 2 c_3 + 12 c_4\right) x^3 + \left(r^2 c_5 - r c_4 + 9 r c_5 - 3 c_4 + 20 c_5\right) x^4 + \left(-r c_5 - 4 c_5\right) x^5 = 0 \quad \textbf{(2.13)}$$

$subs(r = 0, \%)$

$$c_0 + 2 c_2 x + \left(-c_2 + 6 c_3\right) x^2 + \left(-2 c_3 + 12 c_4\right) x^3 + \left(-3 c_4 + 20 c_5\right) x^4 - 4 c_5 x^5 = 0 \quad \textbf{(2.14)}$$

This tells us that c0 and c2 must be zero and hence all the terms that come after c2. c1 is undetermined and unrestricted,
and so $c_1 x = y_1$ is our first solution, and the reduction-of-order formula gives us $y_2(x)$.

We write the solution, step by step:

$$\int \frac{\exp(x)}{x^2}\,\mathrm{d}x$$

$$-\frac{e^x}{x} - \mathrm{Ei}_1(-x) \quad \textbf{(2.15)}$$

That wasn't particularly useful. So, let's do a term-by-term integration of a Taylor polynomial and call it good!
New command: **series**.

$series(\exp(x), x, 8)$

$$1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{120} x^5 + \frac{1}{720} x^6 + \frac{1}{5040} x^7 + \mathrm{O}\left(x^8\right) \quad \textbf{(2.16)}$$

convert(%, *polynom*)

$$1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\frac{1}{720}x^6+\frac{1}{5040}x^7 \quad \textbf{(2.17)}$$

$\frac{\%}{x^2}$

$$\frac{1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\frac{1}{720}x^6+\frac{1}{5040}x^7}{x^2} \quad \textbf{(2.18)}$$

expand(%)

$$\frac{1}{2}+\frac{1}{x^2}+\frac{1}{x}+\frac{x}{6}+\frac{x^2}{24}+\frac{x^3}{120}+\frac{x^4}{720}+\frac{x^5}{5040} \quad \textbf{(2.19)}$$

$$\frac{x^5}{5040}+\frac{x^4}{720}+\frac{x^3}{120}+\frac{x^2}{24}+\frac{x}{6}+\frac{1}{2}+\frac{1}{x}+\frac{1}{x^2} \quad \textbf{(2.20)}$$

sort(%, *x*, *ascending*)

$$\frac{1}{2}+\frac{1}{x^2}+\frac{1}{x}+\frac{x}{6}+\frac{x^2}{24}+\frac{x^3}{120}+\frac{x^4}{720}+\frac{x^5}{5040} \quad \textbf{(2.21)}$$

This gives us our integrand in the Reduction-of-order formula. Now we integrate it. This is $\int \frac{e^{-\int P(x)\,dx}}{y_1(x)^2}\,dx$:

$$\int\left(\frac{1}{2}+\frac{1}{x^2}+\frac{1}{x}+\frac{x}{6}+\frac{x^2}{24}+\frac{x^3}{120}+\frac{x^4}{720}+\frac{x^5}{5040}\right)dx$$

$$\frac{x^6}{30240}+\frac{x^5}{3600}+\frac{x^4}{480}+\frac{x^3}{72}+\frac{x^2}{12}+\frac{x}{2}-\frac{1}{x}+\ln(x) \quad \textbf{(2.22)}$$

The finishing touch is to multiply by $y_1(x)=x$:

$x \cdot \%$

$$\left(\frac{x^6}{30240}+\frac{x^5}{3600}+\frac{x^4}{480}+\frac{x^3}{72}+\frac{x^2}{12}+\frac{x}{2}-\frac{1}{x}+\ln(x)\right)x \quad \textbf{(2.23)}$$

expand(%)

$$-1+\ln(x)\,x+\frac{x^2}{2}+\frac{x^3}{12}+\frac{x^4}{72}+\frac{x^5}{480}+\frac{x^6}{3600}+\frac{x^7}{30240} \quad \textbf{(2.24)}$$

?*quo*
$divide(1, x^2 + 2 \cdot x + 3)$

$$false \quad (1)$$

?*series*