Section 6.3 - Power Series about Singular Points

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \longrightarrow$$
$$y'' + P(x)y' + Q(x)y = 0$$

$x = x_0$ is a singular point if $a_2(x_0) = 0$

Regular: $p(x) = (x-x_0)P(x) = (x-x_0)\dfrac{a_1(x)}{a_2(x)}$ AND

$q(x) = (x-x_0)^2 Q(x) = (x-x_0)^2 \dfrac{a_0(x)}{a_2(x)}$ are both

analytic @ $x = x_0$

Otherwise $x = x_0$ is Irregular Singular Point

Basically, they're saying that $x = x0$ **is a root of multiplicity 1** ***OR*** **a1(x) and a0(x) have factors of x - x0 of sufficiently high multiplicity to cancel out some or all of the x - x0 factors in a2(x).**

We mostly are interested in cases where

a_2, a_1, a_0 share no common factors ("relatively prime")

If that be the case, then "regular" just means that the multiplicity of x = x0 is 1.

By FTA, $a_2(x) = A(x-x_0)^{m_0}(x-x_1)^{m_1}\cdots(x-x_k)^{m_k}$ $\qquad \sum_{i=0}^{k} m_i = n =$ degree of $a_2(x)$

So $P(x) = \dfrac{a_1(x)}{A(x-x_0)^{m_0}\cdots(x-x_k)^{m_k}}$

and $(x-x_0)P(x) = \dfrac{a_1(x)(x-x_0)}{A(x-x_0)^{m_0}\cdots(x-x_k)^{m_k}}$ fails to be analytic

@ $x = x_0$, if $m_0 > 1$, if $a_1(x)$ & $a_2(x)$ share no common factors.

Note: Cauchy-Euler Equations have a regular singular point at $x = 0$.

$a_2x^2y'' + a_1xy' + a_0y = f(x)$

Theorem 6.3.1 Frobenius' Theorem

If $x = x_0$ is a regular singular point of the differential equation (1), then there exists at least one solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n(x - x_0)^n = \sum_{n=0}^{\infty} c_n(x - x_0)^{n+r}, \tag{4}$$

where the number r is a constant to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

The goal is to find r.

If r is not a $\underbrace{\text{nonnegative integer}}_{\text{whole number}}$, then (4) is not a power series.

Example 2 Two Series Solutions

Because $x = 0$ is a regular singular point of the differential equation

$$3xy'' + y' - y = 0, \tag{5}$$

we try to find a solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$.

$x - x_0 = x - 0 \longrightarrow (x-x_0)^r \sum_{n=0}^{\infty} c_n (x-x_0)^n = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r} = y(x)$

Sub into eq'n:

$y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$ &

$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$ $\longrightarrow$

$$3xy''+y'-y = 3x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$= 3\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$= \sum_{n=0}^{\infty}\left(3(n+r)(n+r-1)c_n + (n+r)c_n\right)x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$= x^r\left[\sum_{n=0}^{\infty}\left(3(n+r)(n+r-1)+(n+r)\right)c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n\right]$$

$$= x^r\left[\sum_{n=0}^{\infty}(n+r)(3n+3r-2)c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n\right]$$

Scratch: $(n+r)\left(3(n+r-1)+1\right) = (n+r)(3n+3r-3+1)$

$$= x^r\left[(0+r)(0+3r-2)c_0 x^{-1} + \sum_{n=1}^{\infty}(n+r)(3n+3r-2)c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n\right]$$

$k=n-1$

$$= x^r\left[r(3r-2)c_0 x^{-1} + \sum_{n=0}^{\infty}(n+1+r)(\underbrace{3(n+1)+3r-2}_{3n+3+3r-2})c_{n+1}x^n - \sum_{n=0}^{\infty} c_n x^n\right]$$

$$= x^r\left[r(3r-2)c_0 x^{-1} + \sum_{n=0}^{\infty}\left[(n+r+1)(3n+3r+1)c_{n+1} - c_n\right]x^n\right]$$

We're assuming we've got a power series solution here, because we're at the 200 level and we take Frobenius at his word! There *is* a power series solution of the given form. Therefore, the nexative exponent must go away. Therefore,

$$r(3r-2)=0 \longrightarrow r=0 \text{ OR } r=\frac{2}{3}$$

The Indicial Equation.

Now we sub the two different *r*'s into the last expression, giving us a nice recurrence relations on the cn's.

The book converts to *k* as the index of summation to unify the two series with the same powers of *x*. I just shift the *n*'s around, myself. Never learned it this book's way.

$$Ly = x^r \sum_{n=0}^{\infty} \left[(n+r+1)(3n+3r+1)c_{n+1} - c_n \right] x^n = 0$$

$$\Rightarrow c_{n+1} = \frac{c_n}{(n+r+1)(3n+3r+1)}$$

$r=0$: $(n+0+1)(3n+3(0)+1)c_{n+1} - c_n = 0$

$\Rightarrow \cancel{(n+1)}(3n+1)c_{n+1} = c_n \Rightarrow$

$r=0$: $\boxed{c_{n+1} = \frac{c_n}{(n+1)(3n+1)}}$, $n=0,1,2,\ldots$ $\to r=0$

$r=\frac{2}{3}$ $\quad c_{n+1} = \frac{c_n}{(n+\frac{2}{3}+1)(3n+3(\frac{2}{3})+1)}$

$= \frac{c_n}{(n+\frac{5}{3})(3n+3)}$ oR $\boxed{\frac{c_n}{(3n+5)(n+1)} = c_{n+1}}$, $n=0,1,2,\ldots$ $\to r=\frac{2}{3}$

Both of these converge, by Ratio Test. (6.1)

$r=\frac{2}{3}$: $\frac{c_n}{(3n+5)(n+1)}$, $n=0,1,2,\dots$

$c_1 = \frac{c_0}{5}$

$c_2 = \frac{c_1}{8(2)} = \frac{1}{16}c_1 = \left(\frac{1}{16}\right)\left(\frac{c_0}{5}\right) = \frac{1}{80}c_0 = \frac{1}{2^4 \cdot 5}c_0 = \frac{1}{2!\,8\cdot 5}$

$c_3 = \frac{c_2}{11(3)} = \frac{1}{33}\left(\frac{1}{80}c_0\right) = \frac{1}{11\cdot 3\cdot 80}c_0 = \frac{1}{11\cdot 3\cdot 2^4\cdot 5}c_0 = \frac{1}{11\cdot 8\cdot 3\cdot 2\cdot 5} = \frac{1}{(11\cdot 8\cdot 5)(3!)}$

$c_4 = \frac{c_3}{14\cdot 4} = \left(\frac{1}{2\cdot 7\cdot 2^2}\right)\left(\frac{1}{11\cdot 3\cdot 2^4\cdot 5}\right)c_0 = \frac{1}{14\cdot 11\cdot 8\cdot 5\cdot 4\cdot 3\cdot 2} = \frac{1}{(14\cdot 11\cdot 8\cdot 5)(4!)}$

$y_1 = x^{\frac{2}{3}}\sum_{n=0}^{\infty}\frac{1}{(n!)(5\cdot 8\cdot 11\cdots(3n+2))}x^n$

$n=4$: $14 = 3n+2$ so $\frac{c_0}{n!\,(5\cdot 8\cdot 11\cdots(3n+2))}$

$r=0$: $c_{n+1} = \frac{c_n}{(n+1)(3n+1)}$, $n=0,1,2,\dots$

$c_1 = \frac{c_0}{1} = c_0 = \frac{c_0}{(1)(1)} = \frac{c_0}{(1)(1!)}$ $\quad 1 = 3(1)-2$ $\quad ((3n-2)\cdots(7)(4)(1)$

$c_2 = \frac{c_1}{2(4)} = \frac{c_0}{2\cdot 4} = \frac{c_0}{(4)(2)} = \frac{c_0}{4(2!)}$ $\quad 4 = 3(2)-2$ ✓

$c_3 = \frac{c_2}{3\cdot 7} = \frac{c_0}{(3\cdot 7)(2\cdot 4)} = \frac{c_0}{(7\cdot 4)(3\cdot 2)} = \frac{c_0}{(7\cdot 4)(3!)}$ $\quad 7 = 3(3)-2$

$c_4 = \frac{c_3}{4\cdot 10} = \frac{c_0}{(4\cdot 10)(3\cdot 7)(2\cdot 4)} = \frac{c_0}{(4\cdot 3\cdot 2)(10\cdot 7\cdot 4)} = \frac{c_0}{(4!)(10\cdot 7\cdot 4)}$

$10 = 3(4)-2$

$c_n = \frac{c_0}{(n!)(1\cdot 4\cdot 7\cdot 10\cdots(3n-2))}$

$$\boxed{y_2 = x^0\sum_{n=0}^{\infty}\frac{1}{(n!)(1\cdot 4\cdot 7\cdots(3n-2))}x^n}$$

Is there a way to obtain the indicial equation without all the drama?

(1) $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

$y'' + \frac{a_1(x)}{a_2(x)} y' + \frac{a_0(x)}{a_2(x)} y = 0$ is now of the form

(2) $y'' + P(x)y' + Q(x)y = 0$ P, Q will reveal singular points.

If $x=0$ is regular singular point, then

$p(x) = xP(x)$ & $q(x) = x^2Q(x)$ are analytic @ $x=0$.

This means $\exists$ power series representations:

(12) $p(x) = a_0 + a_1x + a_2x^2 + \ldots$ and $q(x) = b_0 + b_1x + b_2x^2 + \ldots$

Multiply (2) by x^2:

$x^2y'' + x^2P(x)y' + x^2Q(x)y$

(13) $= x^2y'' + x(xP(x))y' + (x^2Q(x))y$

$= x^2y'' + xp(x)y' + q(x)y = 0$

$p(x) = xP(x) = x\frac{a_1(x)}{a_2(x)}$

$q(x) = x^2Q(x) = x^2\frac{a_0(x)}{a_2(x)}$

After substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ and the two series in (12) into (13) and carrying out the multiplication of series, we find the general indicial equation to be

(14) $$r(r-1) + a_0r + b_0 = 0, \tag{14}$$

I think you can prove this with Maple, by truncating all the power series down to their first two or three terms, because (recall), the Indicial Equation boils down to the first term of Ly.

The lowest-degree term is the x^r term:

$$x^r r^2 c_0 + c_0 x^r r a_0 - c_0 x^r r + x^r b_0 c_0$$

$= c_0x^r[r^2 + a_0r - r + b_0] = 0 \Longrightarrow$

$r^2 - r + a_0r + b_0 = 0 \Longrightarrow$

$r(r-1) + a_0r + b_0 = 0$ is the book result.

From now on, we can just have the above indicial equation on a cheat sheet!

Let's see what they mean with Exercise #3

The point $x = 0$ is a regular singular point of the given differential equation.

3. $xy'' + y' + 19y = 0$ $\Rightarrow y'' + \frac{1}{x}y' + \frac{19}{x}y = 0$

Use the general form of the indicial equation $P(x) = \frac{1}{x} \Rightarrow p(x) = xP(x) = \boxed{1 = a_0!}$

14 $r(r-1) + a_0 r + b_0 = 0$ $Q(x) = \frac{19}{x} \Rightarrow q(x) = x^2 Q(x) = 19x$

$\Rightarrow b_0 = 0$ (No constant)

to find the indicial roots of the singularity. (List the indicial roots below as a comma-separated list.)

$\therefore$ I.E. is $r(r-1) + r = 0 \Rightarrow$

$r^2 - r + r = 0 \Rightarrow$

$r^2 = 0 \Rightarrow 0$ is root of multiplicity 2.

ANS: $r = \boxed{0, 0}$

Since they're the same, we only expect one power-series solution.

The point $x = 0$ is a regular singular point of the given differential equation.

8. $xy'' - xy' + y = 0 \implies y'' - y' + \frac{1}{x}y$

$P(x) = -1$, $Q(x) = \frac{1}{x}$

Substitute $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collect terms to rewrite as a single power series.

$p(x) = x P(x) = -x \qquad a_0 = 0$

$q(x) = x^2 Q(x) = x \qquad b_0 = 0$

I.E. : $r(r-1) + a_0 r + b_0 = r(r-1) \overset{\text{set}}{=} 0 \implies r_1 = 1, r_2 = 0$

They differ by an integer. This is Case ii

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}$$

$$\left(\sum_{n=0}^{\infty} c_n x^{n-1+r}(n+r)(n-1+r)\right) + \sum_{n=0}^{\infty}\left(c_n x^{n+r} - c_n x^{n+r}(n+r)\right)$$

$$= x^r \left[\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n-1} + \sum_{n=0}^{\infty}(1-(n+r))c_n x^n\right] = 0 \implies$$

$$r(r-1)c_0 x^{-1} + \sum_{n=1}^{\infty}(n+r)(n+r-1)c_n x^{n-1} + \sum_{n=0}^{\infty}(1-(n+r))c_n x^n$$

$$r(r-1)c_0 x^{-1} + \sum_{n=0}^{\infty}((n+1)+r)((n+1)+r-1)c_{n+1}x^n + \sum_{n=0}^{\infty}(1-(n+r))c_n x^n$$

$$r(r-1)c_0 x^{-1} + \sum_{n=0}^{\infty}\left[((n+1)+r)((n+1)+r-1)c_{n+1} + (-n-r+1)c_n\right]x^n = 0$$

$$\implies c_{n+1} = \frac{-(n+r-1)c_n}{(n+r+1)(n+r)}$$

$$c_{n+1} = \frac{-(n+r-1)c_n}{(n+r+1)(n+r)}$$

Subbing this into the equation, with $r_1 = 1$ gives

$r_1 = 1$:

$$c_{n+1} = \frac{-(n+1-1)c_n}{(n+1+1)(n+1)} = \frac{-n\,c_n}{(n+2)(n+1)}$$

$c_1 = 0$?! This makes all coeffcients after it be zero, so constant soln.?!

Hmmm. Let's look at that sum, again. I'm not seeing the x^0 term.

$$r(r-1)c_0 x^{-1} + \sum_{n=0}^{\infty}\left[(n+r+1)(n+r)c_{n+1} - (n+r-1)c_n\right]x^n$$

setting $y = \sum_{n=0}^{5} c_n x^{n+r}$

$$r^2 c_1 - c_0 r + r c_1 + c_0 + \frac{r^2 c_0 - c_0 r}{x} + \left(r^2 c_2 - r c_1 + 3 r c_2 + 2 c_2\right) x + \left(r^2 c_3 - r c_2 + 5 r c_3 - c_2 + 6 c_3\right) x^2$$
$$+ \left(r^2 c_4 - r c_3 + 7 r c_4 - 2 c_3 + 12 c_4\right) x^3 + \left(r^2 c_5 - r c_4 + 9 r c_5 - 3 c_4 + 20 c_5\right) x^4 + \left(-r c_5 - 4 c_5\right) x^5 = 0$$

I.E. $r(r-1) = 0$

subbing $r = 0$:

$$c_0 + 2 c_2 x + \left(-c_2 + 6 c_3\right) x^2 + \left(-2 c_3 + 12 c_4\right) x^3 + \left(-3 c_4 + 20 c_5\right) x^4 - 4 c_5 x^5 = 0$$

$\Rightarrow$

$c_0 = c_2 = c_3 = \cdots = 0$, so, $y_1 = c_1 x$

So, how do we find y_2?

How about Reduction-of-Order Formula?

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1(x)^2}\,dx$$

$\Rightarrow$ $y'' - y' + \frac{1}{x}y = 0$ $\Rightarrow$ $\int P(x)dx = \int -1\,dx = -x + c$. Let $c = 0$

$P(x) = -1$, $Q(x) = \frac{1}{x}$

$$e^{-\int P(x)dx} = e^x$$

$y_2(x) = c_1 x \int \frac{e^x}{c_1^2 x^2}\,dx$. Don't think we need c_1 in here.

$y_2(x) = x\int \frac{e^x}{x^2}\,dx$ is not bad, if you know "series" command in Maple. We only need a finite # of terms:

series$(\exp(x), x, 8)$

$$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \mathrm{O}(x^8)$$

convert$(\%, polynom)$

$$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7$$

$\frac{\%}{x^2}$

$$\frac{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7}{x^2}$$

expand$(\%)$

$$\frac{x^5}{5040} + \frac{x^4}{720} + \frac{x^3}{120} + \frac{x^2}{24} + \frac{x}{6} + \frac{1}{2} + \frac{1}{x} + \frac{1}{x^2}$$

sort(%, *x*, *ascending*)

$$\frac{1}{2}+\frac{1}{x^2}+\frac{1}{x}+\frac{x}{6}+\frac{x^2}{24}+\frac{x^3}{120}+\frac{x^4}{720}+\frac{x^5}{5040}$$

The above is the first 8 terms of $\frac{e^x}{x^2}$, which we integrate:

$$\int \frac{e^{-\int P(x)\,dx}}{x^2}\,dx = \int \frac{e^{-(-x)}}{x^2}\,dx = \int \frac{e^x\,dx}{x^2}$$

$$\int\left(\frac{1}{2}+\frac{1}{x^2}+\frac{1}{x}+\frac{x}{6}+\frac{x^2}{24}+\frac{x^3}{120}+\frac{x^4}{720}+\frac{x^5}{5040}\right)dx$$

$$\int\left(\frac{1}{2}+\frac{1}{x^2}+\frac{1}{x}+\frac{x}{6}+\frac{x^2}{24}+\frac{x^3}{120}+\frac{x^4}{720}+\frac{x^5}{5040}\right)dx$$

$$\frac{x^6}{30240}+\frac{x^5}{3600}+\frac{x^4}{480}+\frac{x^3}{72}+\frac{x^2}{12}+\frac{x}{2}-\frac{1}{x}+\ln(x)$$

and $y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1(x)^2}\,dx = x\left(\quad \text{ABOVE} \right) =$

$x\cdot\%$

$$= \left(\frac{x^6}{30240}+\frac{x^5}{3600}+\frac{x^4}{480}+\frac{x^3}{72}+\frac{x^2}{12}+\frac{x}{2}-\frac{1}{x}+\ln(x)\right)x$$

expand(%)

$$= -1+\ln(x)\,x+\frac{x^2}{2}+\frac{x^3}{12}+\frac{x^4}{72}+\frac{x^5}{480}+\frac{x^6}{3600}+\frac{x^7}{30240}$$

$$\boxed{y_2 = x\ln(x) - 1 + \frac{x^2}{2}+\frac{x^3}{12}+\frac{x^4}{72}+\cdots}$$

Earlier (failed?) attempt:

$$2\left(\sum_{n=0}^{\infty} c_n x^n n\right) + \sum_{n=0}^{\infty}\left(c_n x^n n^2 - c_n x^n n\right) - x\left(\sum_{n=0}^{\infty} c_n x^n n\right) = 0$$

$$\sum_{n=0}^{\infty}\left(c_n x^n n + c_n x^n n^2\right) - \left(\sum_{n=0}^{\infty} c_n x^{n+1} n\right) = 0 \Rightarrow$$

$c_n x^n (n+n^2)$

$$= \sum_{n=0}^{\infty} (n)(n+1) c_n x^n - \sum_{n=1}^{\infty} (n-1) c_{n-1} x^n = 0 \Rightarrow$$

$$= 0(0+1)c_0 x^0 + \sum_{n=1}^{\infty} n(n+1) c_n x^n - \sum_{n=1}^{\infty} (n-1) c_{n-1} x^n = 0 \Rightarrow$$

$$\sum_{n\geq 1}^{\infty}\left[n(n+1)c_n - (n-1)c_{n-1}\right] x^n = 0$$

$$\Rightarrow \quad n(n+1)c_n = (n-1)c_{n-1} \Rightarrow$$

$$c_n = \frac{(n-1)c_{n-1}}{n(n+1)}$$