Consider the following differential equation to be solved using a power series.

2. $y'' + xy = 0$

Using the substitution $y = \sum_{n=0}^{\infty} c_n x^n$, find an expression for c_{k+2} in terms of c_{k-1} for $k = 1, 2, 3$.

$$y = c_0 + c_1 x + c_2 x^2 + \cdots$$

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots = \sum_{n=0}^{\infty} c_{n+1}(n+1)x^n$$

$$y'' = 2c_2 + 3\cdot 2 c_3 x + 4\cdot 3 c_4 x^2 + \cdots = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$$

$$y'' + xy = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n + x\sum_{n=0}^{\infty} c_n x^n$$

$$= \left(2c_2 + 3\cdot 2c_3 x + 4\cdot 3 c_4 x^2 + 5\cdot 4 c_5 x^3 + \cdots\right. + x\left[c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots\right]$$

$$= \left(2c_2 + 3\cdot 2c_3 x + 4\cdot 3 c_4 x^2 + 5\cdot 4 c_5 x^3 + \cdots\right) + c_0 x + c_1 x^2 + c_2 x^3 + c_3 x^4 + c_4 x^5 + \cdots$$

$$= \sum_{n=0}^{\infty} c_{n+1}(n+1)x^n + \sum_{n=0}^{\infty} c_n x^{n+1}$$

Maple work:

$$y := x \to \sum_{k=0}^{8} \left(c[k] \cdot x^k \right) \qquad y := x \mapsto \sum_{k=0}^{8} c_k \cdot x^k$$

$$yp := \mathrm{D}(y)$$

$$yp := x \mapsto 8 \cdot x^7 \cdot c_8 + 7 \cdot x^6 \cdot c_7 + 6 \cdot x^5 \cdot c_6 + 5 \cdot x^4 \cdot c_5 + 4 \cdot x^3 \cdot c_4 + 3 \cdot x^2 \cdot c_3 + 2 \cdot x \cdot c_2 + c_1$$

$$ypp := \mathrm{D}(yp)$$

$$ypp := x \mapsto 56 \cdot x^6 \cdot c_8 + 42 \cdot x^5 \cdot c_7 + 30 \cdot x^4 \cdot c_6 + 20 \cdot x^3 \cdot c_5 + 12 \cdot x^2 \cdot c_4 + 6 \cdot x \cdot c_3 + 2 \cdot c_2$$

$$ypp(x) + x \cdot y(x)$$

$$56 x^6 c_8 + 42 x^5 c_7 + 30 x^4 c_6 + 20 x^3 c_5 + 12 x^2 c_4 + 6 x c_3 + 2 c_2 + x \left(c_8 x^8 + c_7 x^7 + c_6 x^6 + c_5 x^5 + c_4 x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0 \right)$$

$$expand(\%)$$

$$x^9 c_8 + x^8 c_7 + x^7 c_6 + x^6 c_5 + 56 x^6 c_8 + x^5 c_4 + 42 x^5 c_7 + x^4 c_3 + 30 x^4 c_6 + x^3 c_2 + 20 x^3 c_5 + x^2 c_1 + 12 x^2 c_4 + x c_0 + 6 x c_3 + 2 c_2$$

This helps you see it, better:

$$sort(\%, x, ascending)$$

$$2 c_2 + c_0 x + 6 c_3 x + c_1 x^2 + 12 c_4 x^2 + c_2 x^3 + 20 c_5 x^3 + c_3 x^4 + 30 c_6 x^4 + c_4 x^5 + 42 c_7 x^5 + c_5 x^6 + 56 c_8 x^6 + c_6 x^7 + c_7 x^8 + c_8 x^9$$

This *really* helps you see it, better:

$collect(\%, x)$

$$c_8 x^9 + c_7 x^8 + c_6 x^7 + (c_5 + 56 c_8) x^6 + (c_4 + 42 c_7) x^5 + (c_3 + 30 c_6) x^4 + (c_2 + 20 c_5) x^3 + (c_1 + 12 c_4) x^2 + (c_0 + 6 c_3) x + 2 c_2$$

Even better:

$sort(\%, x, ascending)$

$$2 c_2 + (c_0 + 6 c_3) x + (c_1 + 12 c_4) x^2 + (c_2 + 20 c_5) x^3 + (c_3 + 30 c_6) x^4 + (c_4 + 42 c_7) x^5 + (c_5 + 56 c_8) x^6 + c_6 x^7 + c_7 x^8 + c_8 x^9$$

Nice job of collecting the same-degree terms and factoring out the powers of *x*. I can see it a lot more clearly (and confidently). Don't much care for their step-by-step, but I do observe that it's doing a c[n] and a c[n + 3] in each term. (or c[n-1] and a c[n+2] as the book says).

Book way:

$n=0: \quad x^0$

$c_2 = 0$

$c_0 + 6c_3 = 0 \Longrightarrow c_3 = -\frac{1}{6}c_0$ $\quad n=1: \quad c_{n+2} = -\frac{1}{6}c_{n-1} = -\frac{1}{3\cdot 2}c_{n-1}$ $\quad c_2 = 0$

* $c_1 + 12c_4 = 0 \longrightarrow c_4 = -\frac{1}{12}c_1$ $\quad n=2: \quad c_{n+2} = -\frac{1}{12}c_{n-1} = -\frac{1}{4\cdot 3}c_{n-1}$

* $c_2 + 20c_5 = 0 \Longrightarrow c_5 = -\frac{1}{20}c_2 =$ $\quad n=3: \quad c_{n+2} = -\frac{1}{20}c_{n-1} = -\frac{1}{5\cdot 4}c_{n-1} = 0 \quad c_5 = 0$

$c_3 + 30c_6 = 0 \Longrightarrow c_6 = -\frac{1}{30}c_3 = -\frac{1}{30}\cdot\left(-\frac{1}{6}\right)c_0$ $\quad \left(-\frac{1}{n+2}\right)\left(\frac{1}{n+1}\right)$

* $c_4 + 42c_7 = 0 \Longrightarrow c_7 = -\frac{1}{42}c_4 = -\frac{1}{42}\left(-\frac{1}{12}c_1\right)$ $\quad c_{n+2} = \frac{-1}{(n+2)(n+1)}c_{n-1}$

* $c_5 + 56c_8 = 0 \longrightarrow c_8 = 0$ $\quad c_{k+2} = \frac{-1}{(k+2)(k+1)}c_{k-1}$

$c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7$

$c_0 + c_1x + 0 - \frac{1}{6}c_0x^3 - \frac{1}{12}c_1x^4 + 0 + \left(\frac{1}{30}\right)\left(\frac{1}{6}\right)c_0x^6 + \left(\frac{1}{42}\right)\left(\frac{1}{12}\right)c_1x^7$

$y_1(x) = c_0 - \frac{1}{6}c_0x^3 + \left(\frac{1}{30}\right)\left(\frac{1}{6}\right)c_0x^6 + \ldots = c_0\left(1 - \frac{1}{6}x^3 + \left(\frac{1}{30}\right)\left(\frac{1}{6}\right)x^6 + \ldots\right)$

$y_2(x) = c_1x - \frac{1}{12}c_1x^4 + \frac{1}{42}\left(\frac{1}{12}\right)c_1x^7 + \ldots = c_1\left(x - \frac{1}{12}x^4 + \frac{1}{42}\left(\frac{1}{12}\right)x^7 + \ldots\right)$

$y = c_0y_1 + c_1y_2$