Taking you all the way back to variation of parameters:

$Ly = y'' + P(x)y' + Q(x)y = f(x)$ s.t. $y(x_0) = y_0$, $y'(x_0) = y_0'$

Find linearly independent y_1, y_2 that solve $Ly = 0$,
subject to the IC's.

Find $u_1, u_2 \ni u_1 y_1 + u_2 y_2$ solve $Ly = f(x)$ s.t. $y(x_0) = y'(x_0) = 0$.

write sol'n as the integral $\int_{x_0}^{x} G(x,t) f(t)\, dt$

Now we're looking at Boundary Value Problems on $[a,b]$

$Ly = y'' + P(x)y' + Q(x)y = f(x)$ s.t.

$A_1 y(a) + B_1 y'(a) = 0$

$A_2 y(b) + B_2 y'(b) = 0$

We assume that the corresponding

$Ly = 0$ with the same BC's has only the trivial sol'n
$y \equiv 0$

I think this is much easier to put together if you dive deep into the "Basic Linear Algebra" stuff I posted a while back. BVPs illustrate how cook-booking the Green's Function too much can be a dead end.

Going all the way back:

We have solns y_1, y_2 to $Ly=0 \ni y(x_0)=y_0, y'(x_0)=y_0'$

we look for $u_1, u_2 \ni$

$y_1u_1' + y_2u_2' = 0$ and

$y_1'u_1' + y_2'u_2' = f(x)$

So $A = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$, $\bar{u}' = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}$, and $\bar{f} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$ gives the

matrix equation $A\bar{u}' = \bar{f}$ and the solution is given by

$\bar{u}' = A^{-1}\bar{f}$.

Now, $A^{-1} = \frac{1}{|A|}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}$ and $\bar{u}' = A^{-1}\bar{f}$

$= \frac{1}{|A|}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}\begin{bmatrix} 0 \\ f(x) \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} -y_2 f(x) \\ y_1 f(x) \end{bmatrix} = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}$

$y_1y_2' - y_1'y_2$

$$u_1' = \frac{1}{|A|}(-y_2 f) = \frac{w_1}{w}$$, since $w = |A| =$

$$u_2' = \frac{1}{|A|}(y_1 f) = \frac{w_2}{w}$$

Then $y_p = u_1 y_1 + u_2 y_2 = y_1 \int \frac{-y_2 f}{w}\,dx + y_2(x) \int \frac{y_1 f}{w}\,dx$

or $y_p = y_1(x) \int \frac{w_1}{w}\,dx + y_2(x) \int \frac{w_2}{w}\,dx$

OLD. SKIP?

Green's : $\int_{x_0}^{x} \frac{y_1(x) w_1(t) + y_2(x) w_2(t)}{f(t)\, w(t)} \cdot f(t)\,dt$

$$= \int_{x_0}^{x} \frac{y_1(x)(-y_2(t) f(t)) + y_2(x)(y_1(t) f(t))}{f(t) w(t)} \cdot f(t)\,dt$$

$$= \int_{x_0}^{x} \frac{-y_1(x) y_2(t) + y_2(x) y_1(t)}{w(t)} f(t)\,dt$$

END OF OLD.

I would never think of this in a million years, but go back to...

$u_1(x) = \int \frac{w_1(x)}{w(x)}\,dx$ Integrate from b to x:

$u_1(x) = \int_b^x \frac{w_1(t)}{w(t)}\,dt$

For $u_2(x)$, integrate from a to x:

$u_2(x) = \int_a^x \frac{w_2(t)}{w(t)}\,dt$

Then $u_1 y_1 + u_2 y_2$ is:

$$y_p = y_1(x)\int_b^x \frac{-y_2 f(t)}{w(t)}\,dt + y_2(x)\int_a^x \frac{y_1 f(t)}{w(t)}\,dt$$

$$= y_1(x)\int_x^b \frac{y_2 f(t)}{w(t)}\,dt + y_2(x)\int_a^x \frac{y_1 f(t)}{w(t)}\,dt$$

$$= \int_x^b \frac{y_1(x) y_2(t) f(t)}{w(t)}\,dt + \int_a^x \frac{y_2(x) y_1(t) f(t)}{w(t)}\,dt$$

$$= \int_a^b G(x,t) f(t)\,dt, \text{ where}$$

$$G(x,t) = \begin{cases} \dfrac{y_2(x) y_1(t)}{w(t)} & \text{if } a \le t \le x \\ \dfrac{y_1(x) y_2(t)}{w(t)} & \text{if } x \le t \le b \end{cases}$$

Proceed as in Examples 7 and 8 to find a solution of the given boundary-value problem.

9. $y'' - y' = e^{2x}$, $y(0) = 0$, $y(1) = 0$

$y(x) =$ ✖

$y'' - y' = e^{2x}$

$y'' - y' = 0$ for y_h

$r^2 - r = r(r-1)$, $r = 0, 1$

$\boxed{y_1 = e^{0x},\ y_2 = e^x}$

$\boxed{y_1 = 1}$

$\boxed{y_h = c_1 + c_2 e^x}$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix} = \boxed{e^x = W(x)}$$

$$G(x,t) = \begin{cases} \dfrac{y_2(x) y_1(t)}{W(t)} & \text{if } a \le t \le x \\ \dfrac{y_1(x) y_2(t)}{W(t)} & \text{if } x \le t \le b \end{cases}$$

$\int_a^x$

$\int_b^x = -\int_x^b$

$$= \begin{cases} \dfrac{e^x}{e^t} & \text{if } 0 \le t \le x \\ \dfrac{e^t}{e^t} & \text{if } x \le t \le 1 \end{cases} = \begin{cases} e^{x-t} & \text{if } 0 \le t \le x \\ 1 & \text{if } x \le t \le b \end{cases}$$

x

$$y_p = \int_x^1 e^{2t}\,dt + \int_0^x e^{x-t} e^{2t}\,dt$$

$$= \frac{1}{2}e^{2t}\Big]_x^1 + \int_0^x e^{x+t}\,dt$$

Remember: $y_1 = 1$ (c_1)
$y_2 = e^x$ $(c_2 e^x)$

$$= \frac{1}{2}e^2 - \frac{1}{2}e^{2x} + e^{x+t}\Big]_0^x = \frac{1}{2}e^2 - \frac{1}{2}e^{2x} + e^{2x} - e^x$$

$\frac{1}{2}e^2$: c_1! $\quad -e^x$: c_2!

$$\boxed{y_p = \frac{1}{2}e^{2x}}$$

$$y = y_h + y_p = c_1 + c_2 e^x + \frac{e^{2x}}{2}$$

Now, incorporate the B.C.'s.

$$y(0) = c_1 + c_2 + \frac{1}{2} = 0 \longrightarrow c_1 + c_2 = -\frac{1}{2}$$

$$y(1) = c_1 + c_2 e + \frac{e^2}{2} = 0 \longrightarrow c_1 + e c_2 = -\frac{e^2}{2}$$

$$\left[\begin{array}{cc|c} 1 & 1 & -\frac{1}{2} \\ 1 & e & -\frac{e^2}{2} \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{e}{2} \\ 0 & 1 & -\frac{e}{2} - \frac{1}{2} \end{array}\right]$$

$$y(x) = \frac{e}{2} + \left(-\frac{e}{2} - \frac{1}{2}\right)e^x + \frac{e^{2x}}{2}$$

$$= \boxed{\frac{e}{2} - \frac{e^{x+1}}{2} - \frac{e^x}{2} + \frac{e^{2x}}{2}}$$

$$G(x,t) = \begin{cases} \dfrac{y_2(x)y_1(t)}{w(t)} & \text{if } a \le t \le x \\ \dfrac{y_1(x)y_2(t)}{w(t)} & \text{if } x \le t \le b \end{cases}$$

$$y_h = c_1 + c_2 e^x$$

$$y_1 = 1,\quad y_2 = e^x,\quad W(y_1, y_2) = \boxed{e^x = W(x)}$$

$$y_1 u_1 = \int_b^x \frac{y_1(x)y_2(t)}{w(t)} f(t)\,dt \qquad y_p = y_1(x)\int \frac{w_1}{w} + y_2(x)\int \frac{w_2}{w}$$

$$y_2 u_2 = \int_a^x \frac{y_2(x)y_1(t)}{w(t)} f(t)\,dt$$

$$y_p = y_1 u_1 + y_2 u_2 = \int_x^b \frac{y_1(x)y_2(t)}{w(t)} f(t)\,dt + \int_a^x \frac{y_2(x)y_1(t)}{w(t)} f(t)\,dt$$

$$= \int_0^x \frac{y_1(t)y_2(x)}{w(t)} f(t)\,dt + \int_x^b \frac{y_2(t)y_1(x)}{w(t)} f(t)\,dt = y_p$$

Define

$$G(x,t) = \begin{cases} \dfrac{y_1(t)y_2(x)}{w(t)} & 0 \le t \le x \\ \dfrac{y_2(t)y_1(x)}{w(t)} & x \le t \le b \end{cases}$$

$$y_p = \int_0^1 G(x,t)f(t)\,dt \qquad \text{so what?}$$

For practical purposes, we'll just do the 2 pieces separately.

$$y_p = \int_0^x \frac{e^x}{e^t} \cdot e^{2t}\,dt + \int_x^1 \frac{e^t}{e^t} e^{2t}\,dt$$

$$= e^x \int_0^x e^t\,dt + \int_x^1 e^{2t}\,dt$$

$$= e^x \left[e^t\right]_0^x + \frac{1}{2}e^{2t}\Big]_x^1 =$$

$$= e^x\left[e^x - 1\right] + \frac{1}{2}\left[e^2 - e^{2x}\right] = e^{2x} - e^x + \frac{e^2}{2} - \frac{e^{2x}}{2} = \frac{e^{2x}}{2} - e^x + \frac{e^2}{2} = y_p$$

$$y_p + y_h = c_1 + c_2 e^x + \frac{e^{2x}}{2} - e^x + \frac{e^2}{2}$$

$$= c_1 + c_2 e^x + \frac{e^{2x}}{2}$$

$$y(0) = c_1 + c_2 + \frac{1}{2} = 0$$

$$y(1) = c_1 + c_2 e + \frac{e^2}{2} = 0$$

$$c_1 + c_2 = -\frac{1}{2}$$

$$c_1 + e c_2 = -\frac{e^2}{2}$$

$$c_2 - e c_2 = \frac{-e^2}{2} + \frac{1}{2} = -\frac{1}{2} + \frac{e^2}{2} = +\frac{1}{2}(e^2 - 1)$$

$$(1-e)\,c_2 = \frac{1}{2}(e^2-1)$$

$$\boxed{c_2 = -\frac{1}{2}(1+e)}$$

$$c_1 + c_2 = c_1 - \frac{1}{2} - \frac{1}{2}e = -\frac{1}{2} \rightarrow$$

$$\boxed{c_1 = \frac{1}{2}e}$$

So $c_1 + c_2 e^x + \frac{e^{2x}}{2}$

$$= \frac{1}{2}e + \left(-\frac{1}{2}(e+1)\right)e^x + \frac{e^{2x}}{2}$$

$$= \frac{1}{2}e - \frac{1}{2}e^{x+1} - \frac{1}{2}e^x + \frac{e^{2x}}{2}$$