Proceed as in Example 1 to find a particular solution $y_p(x)$ of the given differential equation in the integral form $y_p(x) = \int_{x_0}^{x} G(x, t) f(t)\, dt$.

ZILLDIFFEQMODAP12 4.8.004.EP.

1. $25y'' - 10y' + y = f(x)$ $\Longrightarrow y'' - \frac{2}{5}y' + \frac{1}{25}y = \boxed{\frac{1}{25}f(x)}$ The "f(x)" we're working with.

Find the linearly independent solutions of the associated homogeneous equation. (Let c_1 and c_2 equal 1. Enter your answers as a comma-separated list.)

$\{y_1(x), y_2(x)\} = \{ \quad \}$

Write the associated homogeneous ODE in Standard Form. Don't forget the "new $f(x)$" on the RHS

$25y'' - 10y' + y = 0$

$y'' - \frac{10}{25}y' + \frac{1}{25}y = 0$

Find the roots of the auxiliary equation:

$\Rightarrow r^2 - \frac{2}{5}r + \left(\frac{1}{5}\right)^2 = \left(r - \frac{1}{5}\right)^2 = 0 \Longrightarrow r = \frac{1}{5}$, multiplicity 2.

$\boxed{\text{Let } y_1 = e^{\frac{1}{5}x} \text{ \& } y_2 = xe^{\frac{1}{5}x}}$. Then

Then $\{y_1, y_2\}$ is fundamental set of sol'ns and

General Sol'n is $\boxed{y_h = c_1y_1 + c_2y_2}$

Determine the value of the Wronskian of the solution functions.

$W\big(y_1(x), y_2(x)\big) =$

$\boxed{y_1 = e^{\frac{1}{5}x}, \ y_2 = xe^{\frac{1}{5}x}}$

Let $A = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$ Then the Wronskian of y_1, y_2 is

$$W(y_1, y_2) = |A| = \begin{vmatrix} e^{\frac{1}{5}x} & xe^{\frac{1}{5}x} \\ \frac{1}{5}e^{\frac{1}{5}x} & e^{\frac{1}{5}x} + \frac{1}{5}xe^{\frac{1}{5}x} \end{vmatrix} = e^{\frac{2}{5}x} + \frac{1}{5}xe^{\frac{2}{5}x} - \frac{1}{5}xe^{\frac{2}{5}x} = e^{\frac{2}{5}x}$$

$$= y_1y_2' - y_1'y_2$$

$\boxed{W = e^{\frac{2}{5}x}}$

Find a particular solution $y_p(x)$ of the given differential equation in the integral form $y_p(x) = \int_{x_0}^{x} G(x, t) f(t)\, dt$.

$y_p(x) = \int_{x_0}^{x} \left(\quad \right) f(t)\, dt$

$y_1 = e^{\frac{1}{5}x}, y_2 = xe^{\frac{1}{5}x}$

$W = e^{\frac{2}{5}x}$

Book way:

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1y_2' - y_1'y_2}$$

$$= \frac{e^{\frac{1}{5}t}xe^{\frac{1}{5}x} - e^{\frac{1}{5}x}te^{\frac{1}{5}t}}{e^{\frac{1}{5}t}\left(e^{\frac{1}{5}t} + \frac{1}{5}te^{\frac{1}{5}t}\right) - \frac{1}{5}e^{\frac{1}{5}t}te^{\frac{1}{5}t}} = \frac{xe^{\frac{1}{5}(t+x)} - te^{\frac{1}{5}(x+t)}}{e^{\frac{2}{5}t} + \frac{1}{5}te^{\frac{2}{5}t} - \frac{1}{5}te^{\frac{2}{5}t})}$$

$$\frac{(x-t)e^{\frac{1}{5}(x+t)}}{e^{\frac{2}{5}t}} = (x-t)e^{\frac{1}{5}x + \frac{1}{5}t}e^{-\frac{2}{5}t} = (x-t)e^{\frac{1}{5}(x-t)}$$

The only problem with this is it'll be

$$y_p = \int_{x_0}^{x} G(x,t)\frac{f(t)}{25}\, dt$$

, since we divided by 25 to put in standard form. So to get it right:

$$\boxed{G(x,t) = \frac{1}{25}(x-t)e^{\frac{1}{5}(x-t)}}$$

What if you don't trust yourself to regurgitate

$$\frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1y_2' - y_1'y_2}\ ?$$

Put it ON A CHEAT SHEET.

Also good to remember

want $u_1, u_2 \ni u_1y_1 + u_2y_2$ solves $Ly = f(x)$

Then $u_1 = \int \frac{w_1}{w}$ & $u_2 = \int \frac{w_2}{w}$

$$w_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

If you can rember this much, you can build $G(x,t)$ from scratch, with

$$y_p = y_1(x)\int_{x_0}^{x} \frac{w_1(t)}{w(t)}\,dt + y_2(x)\int_{x_0}^{x} \frac{w_2(t)}{w(t)}\,dt$$

What if you can't remember w_1 & w_2?

Remember $A\bar{u}' = \bar{f}$

& $A^{-1} = \frac{1}{A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

so that $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = A^{-1}\begin{bmatrix} 0 \\ f(x) \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} -b f(x) \\ a f(x) \end{bmatrix}$

$u_1 = \int \frac{-bf(x)}{|A|}$ & $u_1y_1 = y_1u_1 = y_1\int \frac{-bf(x)}{|A|}$

$$y_p = y_1(x)\int_{x_0}^{x} \frac{w_1(t)}{w(t)}\,dt + y_2(x)\int_{x_0}^{x} \frac{w_2(t)}{w(t)}\,dt$$

$$= y_1(x)\int_{x_0}^{x} \frac{-y_2(t)f(t)}{w}\,dt + y_2(x)\int_{x_0}^{x} \frac{y_1(t)f(t)}{w}\,dt$$

$$= \int_{x_0}^{x}\left(\frac{-y_1(x)y_2(t)}{w} + \frac{y_2(x)y_1(t)}{w}\right)f(t)\,dt$$

$$= \int_{x_0}^{x} \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{w(t)} f(t)\,dt$$

All goes back to reduction of order.

" " " " Variation of Parameters.

$$\begin{matrix} y_1 & y_2 \\ y_1' & y_2' \end{matrix}$$

$$y_1u_1' + y_2u_2' = 0$$
$$y_1'u_1' + y_2'u_2' = f(x)$$

$$y_1u_1' + y_2u_2' + y_3u_3' = 0$$
$$y_1'u_1' + y_2'u_2' + y_3'u_3' = 0$$
$$y_1''u_1' + y_2''u_2' + y_3''u_3' = f(x)$$

If you look at the form of the solutions, they're only looking for you to write the *G*(*x*,*t*)*f*(*t*)

(2) $y'' + 6y' - 27y = x^2$

$(r+9)(r-3)$ $y_1 = e^{3x}$, $y_2 = e^{-9x}$

$$|A| = \begin{vmatrix} e^{3x} & e^{-9x} \\ 3e^{3x} & -9e^{-9x} \end{vmatrix} = -9e^{-6x} - 3e^{-6x} = \boxed{-12e^{-6x} = W}$$

$$w_1 = \begin{vmatrix} 0 & y_2 \\ x^2 & y_2' \end{vmatrix} = \begin{vmatrix} 0 & e^{-9x} \\ x^2 & -9e^{-9x} \end{vmatrix} = -x^2 e^{-9x}$$

$$w_2 = \begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & x^2 \end{vmatrix} = x^2 e^{3x}$$

$$y_p = \int_{x_0}^{x} \frac{y_1(x)W_1(t) + y_2(x)W_2(t)}{W(t)} = \int_{x_0}^{x} \frac{-t^2 e^{3x} e^{-9t} + t^2 e^{-9x} e^{3t}}{-12e^{-6t}}\, dt$$

$$= \frac{1}{12}\int_{x_0}^{x} \left(e^{9x}e^{-9t} - e^{-3x}e^{3t}\right)t^2\, dt = \frac{1}{12}\int_{x_0}^{x}\left(e^{9(x-t)} - e^{3(t-x)}\right)t^2\, dt$$

What the WebAssign is looking for is the

$\frac{1}{12}\left(e^{9(x-t)} - e^{3(t-x)}\right)t^2$ that goes in the integrand. The $\frac{1}{12}\left(e^{9(x-t)} - e^{3(t-x)}\right)$ can be obtained fairly quickly once you have $W = y_1'y_2 - y_1y_2'$ and then all you need is $\frac{1}{f(t)}\left(y_1(x)W_1(t) + y_2(x)W_2(t)\right) = \frac{1}{f(t)}\left(y_1(x)(-y_2 f(t)) + y_2(x)y_1(t)f(t)\right)$

$$= -y_1(x)y_2(t) + y_1(t)y_2(x)$$

$$= y_1(t)y_2(x) - y_1(x)y_2(t) \;!$$

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)} \text{ is THE}$$

$$y_p = \int_{x_0}^{x} G(x,t)f(t)\,dt$$

The $f(t)$ will be the original $f(t)$, divided by the leading coefficient $a_2(t)$ in original:

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$$

(4) $y'' + 18y' + 81y = x$; $y(0) = y'(0) = 0$

$y_h(x) = c_1 e^{-9x} + c_2 x e^{-9x} = c_1 y_1(x) + c_2 y_2(x)$, by inspection

$$W = \begin{vmatrix} e^{-9x} & xe^{-9x} \\ -9e^{-9x} & e^{-9x} - 9xe^{-9x} \end{vmatrix} = e^{-18x} - 9xe^{-18x} + 9xe^{-18x} = e^{-18x}$$

$$W_1 = \begin{vmatrix} 0 & xe^{-9x} \\ x & e^{-9x} - 9xe^{-9x} \end{vmatrix} = -x^2 e^{-9x} = -xe^{-9x} f(x)$$

$$W_2 = \begin{vmatrix} e^{-9x} & 0 \\ -9e^{-9x} & x \end{vmatrix} = xe^{-9x} = e^{-9x} f(x)$$

GREEN'S FUNCTION:

$$G(x,t) = \frac{y_1(x)W_1(t) + y_2(x)W_2(t)}{f(t)W(t)}$$

$$= \frac{y_1(x)(-y_2(t)f(t)) + y_2(x)(y_1(t)f(t))}{f(t)W(t)}$$

$$= \frac{-y_1(x)y_2(t) + y_2(x)y_1(t)}{W(t)} = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2}$$

I always go back at least as far as the Wronskian *W* and the *W*1 and *W*2 computations. Then I observe that the Green's Function is just

$$\frac{y_1(x)W_1(t) + y_2(x)W_2(t)}{f(t)W(t)},$$

which gets me to the Green's Function

$$\frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}$$

via

$$\frac{y_1(x)(-y_2(t)f(t)) + y_2(x)(y_1(t)f(t))}{f(t)W(t)} = \frac{-y_1(x)y_2(t) + y_2(x)y_1(t)}{W(t)}$$

$$= \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}$$

Book way is to compute all the quantities in the above formula and turn the crank.

The messy denominator is just the Wronskian determinant.

I don't know how far back you should go or how formulaic you should be.

Playing with this for probably 3 pages, I can now regurgitate

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}$$

This business of computing W_1 & W_2 goes back to the system of linear equations

$A\bar{u}' = \bar{f}$, where $A = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$ is Wronskian Matrix &

$\bar{u}' = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}$, $\bar{f} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$

Directly, w/ A^{-1}, we have

$$\bar{u}' = A^{-1}\bar{f} = \frac{1}{|A|}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}\begin{bmatrix} 0 \\ f(x) \end{bmatrix} = \frac{1}{W}\begin{bmatrix} -y_2 f(x) \\ y_1 f(x) \end{bmatrix} = \begin{bmatrix} \frac{-y_2 f(x)}{W} \\ \frac{y_1 f(x)}{W} \end{bmatrix}$$

$$y_p = u_1 y_1 + u_2 y_2 = y_1 u_1 + y_2 u_2 = y_1 \int \frac{W_1}{W} + y_2 \int \frac{W_2}{W}$$

4.8: Solve $Ly_h = 0$ with $y_h(x_0) = y_0$, $y_h'(x_0) = y_0'$.

Solve $Ly_p = f(x)$ with $y_p(x_0) = y_p'(x_0) = 0$

$$\underbrace{y_p(x) - y_p(x_0)}_{0} = y_1(x)\int_{x_0}^{x} \frac{W_1(t)}{W(t)}dt + y_2(x)\int_{x_0}^{x} \frac{W_2(t)}{W(t)}dt$$

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