Section 4.7 - Cauchy-Euler Equations

$$Ly = \sum_{k=0}^{n} a_k x^k \frac{d^k y}{dy^k} = a_0 y + a_1 x y' + a_2 x^2 y'' + \dots + a_k x^k y^{(k)} + \dots + a_n x^n y^{(n)} = f(x)$$

We'll focus on the case $n=2$.

$$a_2 x^2 y'' + a_1 x y' + a_0 y = f(x)$$

$$a x^2 y'' + b x y' + c y = f(x) \quad \text{write in standard form}$$

Cauchy-Euler. Very specialized ODE.

Sub $y = x^m$ into eqn:

$$a x^2 (m(m-1) x^{m-2}) + b x (m x^{m-1}) + c x^m$$

$$Ly = x^m (a(m(m-1)) + bm + c)$$

As before, we solve $Ly = 0$, the homogeneous equation, whose solutions will provide the grist for the particular solutions, via Variation of Parameters.

$$Ly = x^m (a(m(m-1)) + bm + c) = 0 \implies x^m = 0. \text{ No help. OR}$$

$$a(m^2 - m) + bm + c = 0 \implies$$

$$am^2 - am + bm + c = 0 \implies$$

$$am^2 + (b-a)m + c = 0 \quad \text{completing the square chops}$$

$$m^2 + \frac{b-a}{a} m + \frac{c}{a} = m^2 + \frac{b-a}{a} m + \left(\frac{b-a}{2a}\right)^2 - \left(\frac{b-a}{2a}\right)^2 + c \cdot \frac{4a}{4a}$$

$$= \left(m + \frac{b-a}{2a}\right)^2 - \frac{(b-a)^2 -}{4a^2} = 0 \implies$$

$$\left(m + \frac{b-a}{2a}\right)^2 = \frac{(b-a)^2 - 4ac}{4a^2} \implies$$

$$m + \frac{b-a}{2a} = \pm \frac{\sqrt{(b-a)^2 - 4ac}}{2a}$$

$$m = \frac{a - b \pm \sqrt{(b-a)^2 - 4ac}}{2a}$$

3 cases for m:

i. 2 distinct, real zeros m_1 & m_2 $\Longrightarrow$ $y = c_1 x^{m_1} + c_2 x^{m_2}$ Slick

ii. 1 repeated root m $\longrightarrow$ $y = c_1 x^m + c_2 x^m \ln(x)$!

iii. 2 distinct, complex roots. $m = \alpha \pm i\beta$ $\Longrightarrow$ $y = x^{\alpha + i\beta}$ $x^\alpha \underbrace{(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))}_{\text{From } x^{i\beta}}$ $c_1 x^{\alpha+i\beta} + c_2 x^{\alpha - i\beta}$

i. Not much to say. It's just slick.

ii. This places the restriction $(a-b)^2 - 4ac = 0$, which means that $m = \frac{a-b}{2a}$ $\boxed{y_1 = x^m = x^{\frac{a-b}{2a}} \quad \text{Remember!}}$

This is a "Reduction-of order situation

Book Way:

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

Just be <u>sure</u> to use the original eq'n in STANDARD FORM!

$$y'' + \frac{b}{ax}y' + \frac{c}{ax^2}y = 0$$

Identify $P(x) = \frac{b}{ax} \implies \int P(x)\,dx = \frac{b}{a}\ln|x|$ (Let $c=0$)

$x=0$ $\cancel{\to}$, so either $x \in (-\infty, 0)$ or $x \in (0,\infty)$. Choose $(0,\infty)$.
$(-\infty,0)$ is similar.

So $\boxed{\int P(x)\,dx = \frac{b}{a}\ln(x)}$ $\left((-\infty,0), \text{ it'd be } \ln(-x)! \right)$

$$\implies y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}\,dx = x^m \int \frac{e^{-\frac{b}{a}\ln(x)}}{x^{2m}}\,dx =$$

$$= x^m \int \frac{e^{\ln(x^{-\frac{b}{a}})}}{x^{2m}}\,dx = x^m \int \frac{x^{-\frac{b}{a}}}{x^{2m}}\,dx = x^m \int \frac{x^{-\frac{b}{a}}}{x^{2\left(\frac{a-b}{2a}\right)}}\,dx$$

$$= x^m \int x^{-b/a}\, x^{\left(\frac{b-a}{a}\right)}\,dx = x^{\frac{a-b}{2a}} \int \frac{1}{x}\,dx = \boxed{x^m \ln(x) = y_2}$$

$$\boxed{y_h = c_1 y_1 + c_2 y_2 = c_1 x^m + c_2 x^m \ln(x)}$$ Repeated roots boil down to this!

iii. $m = \alpha \pm i\beta$, $\alpha, \beta \in \mathbb{R}$

$$y_{21} = x^{\alpha+i\beta} = x^\alpha x^{i\beta} = x^\alpha e^{\ln(x^{i\beta})} = x^\alpha e^{i\beta \ln(x)} = x^m e^{i(\beta \ln(x))}$$

sol'ns come in conjugate pairs, so $m = \alpha - i\beta$ is a root, & so

$x^{\alpha - i\beta}$ is a sol'n. $\Rightarrow$

$C_1 x^{\alpha+i\beta} + C_2 x^{\alpha-i\beta}$ is gen'l sol'n.

what do these look like? Use $x = e^{\ln(x)}$ trick.

$$x^{\alpha+i\beta} = x^\alpha x^{i\beta} = x^\alpha e^{\ln(x^{i\beta})} = x^\alpha e^{i(\beta\ln(x))} = x^\alpha(\cos(\beta\ln(x)) + i\sin(\beta\ln(x)))$$

$$x^{\alpha-i\beta} = x^\alpha(\cos(\beta\ln(x)) - i\sin(\beta\ln(x)))$$

$x^\alpha\cos(\beta\ln(x))$ & $x^\alpha\sin(\beta\ln(x))$ form a 2-element set, each element of which solves the hom. eqn. Wronskian proves they're linearly independent.

Gen'l Sol'n:

$$y_h = c_1 x^\alpha \cos(\beta\ln(x)) + c_2 x^\alpha \sin(\beta\ln(x))$$

or $x^\alpha(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$

Fundamental Set of sol'ns:

$$\boxed{\begin{aligned} y_1 &= x^\alpha\cos(\beta\ln(x)) \\ y_2 &= x^\alpha\sin(\beta\ln(x)) \end{aligned}}$$

$$ax^2y'' + bxy' + cy = 0$$

$$y = x^{\alpha}\cos(\beta \ln(x))$$

$$y' = \alpha x^{\alpha-1}\cos(\beta(\ln(x))) + x^{\alpha}(-\sin(\beta \ln(x)))\left(\frac{\beta}{x}\right)$$

$$= \alpha x^{\alpha-1}\cos(\beta \ln(x)) - \beta x^{\alpha-1}\sin(\beta \ln(x))$$

$$y'' = \alpha(\alpha-1)x^{\alpha-2}\cos(\beta \ln(x)) - \alpha x^{\alpha-1}(\sin(\beta \ln(x)))\left(\frac{\beta}{x}\right)$$

$$- \beta(\alpha-1)x^{\alpha-2}\sin(\beta \ln(x))$$

I really don't want to plug this in & check by hand.

Meh. Let's at least confirm it with an example

Solve $x^2y'' + 3xy' + 5y = 0$

$a=1, b=3, c=5$

$am^2 + (b-a)m + c$

$= m^2 + 2m + 5 = 0 \rightarrow$

$m^2 + 2m + 1^2 = 1 - 5 = -4$

$(m+1)^2 = -4$

$m = -1 \pm 2i$

$$\left.\begin{array}{l} y_1 = \frac{1}{x}\cos(2\ln(x)) \\ y_2 = \frac{1}{x}\sin(2\ln(x)) \end{array}\right\} y_h = c_1y_1 + c_2y_2$$

Particular Sol'n: Re-write in STANDARD FORM

$$y'' + \frac{b}{ax}y' + \frac{c}{ax^2}y = \frac{f(x)}{ax^2} = g(x)$$

How about $f(x) = \csc(x)$?

We solve $Ly = f(x)$

Wronskian of y_1, y_2:

$$A = \begin{bmatrix} \frac{1}{x}\cos(2\ln(x)) & \frac{1}{x}\sin(2\ln(x)) \\ -\frac{1}{x^2}\cos(2\ln(x)) - \frac{2}{x^2}\sin(2\ln(x)) & -\frac{1}{x^2}\sin(2\ln(x)) + \frac{2}{x^2}\cos(2\ln(x)) \end{bmatrix} \longrightarrow$$

$$= |A| = \left(-\frac{1}{x^3}\sin(2x)\cos(2x)\right) + \frac{2}{x^3}\cos^2(2x)\left(+\frac{1}{x^3}\sin(2x)\cos(2x)\right) + \frac{2}{x^3}\sin^2(2x)$$

$$= \frac{2}{x^3}\left(\sin^2(2x) + \cos^2(2x)\right) = \boxed{\frac{2}{x^3} = W(x)}$$

$$y_{p_1} = y_1\int \frac{W_1}{W}dx$$

$$A\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$W_1 = \begin{vmatrix} 0 & \frac{1}{x}\sin(2x) \\ g(x) & -\frac{1}{x^2}\sin(2x) + \frac{2}{x}\cos(2x) \end{vmatrix} = -\frac{1}{x}\sin(2x)g(x)$$

$$W_2 = \begin{vmatrix} \frac{1}{x}\cos(2x) & 0 \\ \sim & g(x) \end{vmatrix} = \frac{1}{x}\cos(2x)g(x)$$

We write $y_p = y_{p_1} + y_{p_2} = \boxed{y_1\int \frac{W_1}{W}dx + y_2\int \frac{W_2}{W}dx}$

$$= y_1\int \frac{-y_2 g(x)}{W}dx + y_2\int \frac{y_1 g(x)}{W}dx$$

$$= y_1\int \frac{-y_2 g(x)}{y_1y_2' - y_1'y_2}dx + y_2\int \frac{y_1 g(x)}{y_1y_2' - y_1'y_2}$$

Lots of tedium & opportunities to make small mistakes.

Switch to Maple to turn the crank.

4.8 - Green's Function

Recall

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$y_p = y_1 \int \frac{W_1}{W}\,dx + y_2 \int \frac{W_2}{W}\,dx$$

$$= y_1 \int \frac{-y_2 f(x)}{y_1 y_2' - y_1' y_2}\,dx + y_2 \int \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2}\,dx$$

We have an IVP, $y(x_0) = y_0$, $y'(x_0) = y_0'$

Solve $Ly = y'' + P(x)y' + Q(x)y = 0$ with the above initial conditions

Now, we want to solve $Ly = f(x)$ with homogeneous initial conditions!
Assume we have y_1, y_2 in hand for sol'ns to the homogeneous eq'n. Then the particular sol'n looks like the top thing. But let's switch to definite integrals:

$$y_1 \int \frac{-y_2 f(x)}{y_1 y_2' - y_1' y_2}\,dx + y_2 \int \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2}\,dx$$

$$y_p(x) = y_1(x) \int_{x_0}^{x} \frac{W_1(t)}{W(t)}\,dt + y_2(x) \int_{x_0}^{x} \frac{W_2(t)}{W(t)}\,dt$$

$$= y_1(x) \int_{x_0}^{x} \frac{-y_2(t) f(t)}{W(t)}\,dt + y_2(x) \int_{x_0}^{x} \frac{y_1(t) f(t)}{W(t)}\,dt$$

$$= \int_{x_0}^{x} \frac{-y_1(x) y_2(t) f(t)}{W(t)}\,dt + \int_{x_0}^{x} \frac{y_2(x) y_1(t) f(t)}{W(t)}\,dt$$

$$= \int_{x_0}^{x} \frac{y_2(x) y_1(t) f(t) - y_1(x) y_2(t) f(t)}{W(t)}\,dt$$

$$= \int_{x_0}^{x} \frac{y_1(t) y_2(x) - y_1(x) y_2(t)}{W(t)} f(t)\,dt$$

$$= \int_{x_0}^{x} G(x,t) f(t)\,dt, \text{ where } G(x,t) = \frac{y_1(t) y_2(x) - y_1(x) y_2(t)}{y_1(t) y_2'(t) - y_1'(t) y_2(t)}$$

is called the Green's Function

What I do : $G(x,t) = \dfrac{y_1(x) W_1(t)}{f(t) W(t)} + \dfrac{y_2(x) W_2(t)}{f(t) W(t)}$

Comes from $y_p = y_1 \int \frac{W_1}{W} + y_2 \int \frac{W_2}{W}$

$$= y_1(x) \int_{x_0}^{x} \frac{W_1(t)}{W(t)}\,dt + y_2(x) \int \frac{W_2(t)}{W(t)}\,dt.$$