$$\sum_{k=0}^{n} a_k x^k \frac{d^k y}{dy^k} = 0$$

$$\frac{dx}{dt} - 10\frac{dy}{dt} = 66y$$

$$\frac{dx}{dt} = 10\frac{dy}{dt} + 66y$$

2^{nd}-ORDER:

$y = x^m$ subbed

$$a_2 x^2 (m(m-1)x^{m-2}) + a_1 x \cdot m x^{m-1} + a_0 x^m$$

$$a_2 x^2 x^{m-2}(m(m-1)) + a_1 x \, x^{m-1} m + a_0 x^m$$

$$= (a_2 m(m-1) + a_1 m + a_0) x^m$$

$$= (a_2 m^2 - a_2 m + a_1 m + a_0) x^m$$

$$= (a_2 m^2 + (a_1 - a_2) m + a_0) x^m$$

$a = a_2, b = a_1 - a_2, c = a_0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$b^2 - 4ac = (a_1 - a_2)^2 - 4(a_2)(a_0)$$

$$= a_1^2 - 2a_1 a_2 + a_2^2 - 4a_2 a_0$$

$$m = \frac{a_2 - a_1 \pm \sqrt{a_1^2 - 2a_1 a_2 + a_2^2 - 4a_0 a_2}}{2a_2}$$

$$a(x)y'' + b(x)y' + c(x) = f(x)$$

$$\Longrightarrow y'' + P(x)y' + Q(x) = \frac{1}{a(x)}f(x)$$

$$\int x \tan(\ln(x))dx$$

$$\int \frac{1}{x}\tan(\ln(x))dx$$

Matrices and Determinants

A matrix is an array of numbers, like

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, or $C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2×3 $\quad 2\times 2 \quad$ 3×1

Define AC by

$$AC = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (1)(1) + (2)(2) + 3(3) \\ (4)(1) + (5)(2) + 6(3) \end{bmatrix} = \boxed{\begin{bmatrix} 14 \\ 32 \end{bmatrix} = AC}$$

$2\times 3 \quad 3 \times 1 \quad 2 \times 1$

have to be the same

I = Identity (for multiplication) $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$IA = AI = A$ for any 2×2 A.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$I\bar{x} = \bar{x}$ for any $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

what would be nice is if we had multiplicative Inverse to act like $\frac{1}{A}$, so we could solve

$A\bar{x} = \bar{b}$ like we solve $3x = 12$

$\frac{1}{3}\cdot 3x = \frac{1}{3}\cdot 12$ OR $3^{-1}\cdot 3x = 3^{-1}\cdot 12$

$x = 4$

$A^{-1}A\bar{x} = A^{-1}\bar{b}$

$\bar{x} = A^{-1}\bar{b}$. Slick.

we find the elusive A^{-1}

Cramer's Rule for 2×2:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = B$$

Proof:

$$BA = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & bd-bd \\ -ac+ac & ad-bc \end{bmatrix}$$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \Rightarrow BA = I$

& $AB = I$, as well.

$B = A^{-1}$ is "A-inverse."

MatrixInverse(A)

with(LinearAlgebra):

We can solve a system like

$1x + 2y = 3$

$4x + 5y = 6$

. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

Then $A\bar{x} = \bar{b} \Rightarrow$

$A^{-1}A\bar{x} = \bar{x} = A^{-1}\bar{b} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$= \frac{1}{5-8}\begin{bmatrix} 5 & -2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 15-12 \\ -12+6 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 3 \\ -6 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \bar{x}}$

$A\bar{x} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ ✓

$\boxed{\begin{array}{l} x = -1 \\ y = 2 \end{array}}$ is sol'n.

Cramer's Rule, in General

Let $A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \underset{3 \times 3}{\begin{bmatrix} a_{ij} \end{bmatrix}}$

$A_2 = \begin{bmatrix} a_{11} & a_{22} \\ a_{21} & a_{22} \end{bmatrix}$ Then the determinant of A_2 is

$= a_{11}a_{22} - a_{21}a_{12} = |A| = \det(A)$

For Bigger matrices, we expand by minors

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$$

$$= (-1)^{1+1}A_{11} + (-1)^{1+2}A_{12} + (-1)^{1+3}A_{13}$$

$$= C_{11} + C_{12} + C_{12}$$

Now, it turns out that A^{-1} is given by

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$\text{Wronskian}(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ want inverse of $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$

$$= \frac{1}{y_1 y_2' - y_2 y_1'} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} = A^{-1} \quad !$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

4.6 Solve

$$y_1 u_1' + y_2 u_2' = 0$$

$$y_1' u_1' + y_2' u_2' = f(x)$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ f(x) \end{bmatrix} = \frac{1}{W(y_1, y_2)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

3x3's just as easy symbolically

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f(x) \end{bmatrix}$$

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\frac{1}{A}\begin{bmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{bmatrix} = \frac{1}{A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\left|\begin{bmatrix} 1 \end{bmatrix}\right| = 1$$

$|-5| = -5$?!

Yes. if $|*|$ means "determinant"!