§4.6 #6

We solve $2y''+y'-y = x+5$

$$y''+\frac{1}{2}y'-\frac{1}{2}y = \underline{\frac{1}{2}(x+5)}$$

$$y''+P(x)y'+Q(x)y = f(x)$$

Finding y_1, y_2 that solve the hom eq'n.

C.P./A.E.: $2m^2+m-1 = 0$

$$0 = (2m-1)(m+1) \Rightarrow$$

$$\Rightarrow m = \frac{1}{2}, -1$$

$y_1 = e^{\frac{1}{2}x}$, $y_2 = e^{-x}$. Then they form a fundamental set of sol'ns.

Form the Wronski matrix $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = A$

$$= \begin{bmatrix} e^{\frac{1}{2}x} & e^{-x} \\ \frac{1}{2}e^{\frac{1}{2}x} & -e^{-x} \end{bmatrix}$$

$$y_1u_1' + y_2u_2' = 0$$
$$y_1'u_1' + y_2'u_2' = f(x)$$

$A = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$, $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ & $\bar{f} = \begin{bmatrix} 0 \\ \frac{x+5}{2} \end{bmatrix}$

we solve $A\bar{u} = \bar{f} \Rightarrow$

$\bar{u} = A^{-1}\bar{f}$, where

$$A^{-1} = \frac{1}{W}\begin{bmatrix} -e^{-x} & -e^{-x} \\ -\frac{1}{2}e^{\frac{1}{2}x} & e^{\frac{1}{2}x} \end{bmatrix}\begin{bmatrix} 0 \\ \frac{x+5}{2} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= -\frac{2}{3}e^{\frac{1}{2}x}\begin{bmatrix} -e^{-x}\left(\frac{x+5}{2}\right) \\ \left(\frac{x+5}{2}\right)e^{\frac{1}{2}x} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\left(\frac{x+5}{2}\right)e^{-\frac{1}{2}x} \\ -\frac{2}{3}\left(\frac{x+5}{2}\right)e^{x} \end{bmatrix} = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}$$

$$W = \begin{vmatrix} e^{\frac{1}{2}x} & e^{-x} \\ \frac{1}{2}e^{\frac{1}{2}x} & -e^{-x} \end{vmatrix} = -e^{-\frac{1}{2}x} - \frac{1}{2}e^{-\frac{1}{2}x} = -\frac{3}{2}e^{-\frac{1}{2}x}$$

$$\Rightarrow \frac{1}{W} = -\frac{2}{3}e^{\frac{1}{2}x}$$

$$\Rightarrow u_1 = \frac{1}{3}\int (x+5)e^{-\frac{1}{2}x}dx = \frac{-2(x+7)e^{-x/2}}{3}$$

$$u_2 = -\frac{1}{3}\int (x+5)e^{x}dx = \frac{-(x+4)e^{x}}{3}$$

so $y_p = u_1y_1 + u_2y_2 = -\frac{2}{3}(x+7)e^{-\frac{x}{2}}e^{\frac{1}{2}x} - \frac{1}{3}(x+4)e^{x}e^{-x}$

$$= -\frac{2}{3}x - \frac{14}{3} - \frac{1}{3}x - \frac{4}{3}$$

$$= -x - \frac{18}{3} = -x - 6 = y_p$$

$$\therefore \boxed{y = y_c + y_p = c_1e^{\frac{1}{2}x} + c_2e^{-x} - x - 6}$$