

4.3

Characteristic Polynomial

$$y'' - 3y' + 2y = 0$$

$$D^2y - 3Dy + 2y = 0$$

$$(D^2 - 3D + 2)y = 0$$

$$m^2 - 3m + 2 = 0$$

$$\frac{dP}{dt} = kP$$

$$\Rightarrow P = C_1 e^{kt}$$

$$P' - kP = 0$$

$$D P - k P = 0$$

$$(D - k)P = 0 \quad D = k \Rightarrow e^{kt}$$

$$\Rightarrow (m-2)(m-1) = 0 \Rightarrow$$

e^{2x}, e^{1x} are a fundamental set of solns.

Special: Use Euler's Formula

$$\text{for } e^{i\theta} = \cos \theta + i \sin \theta$$

⋮

if $m = bi$ is a soln of C.P.

Then e^{ibx} is a soln of ODE.

If coefficients of the ODE are Real, then complex roots of C.P. will occur in conjugate pairs.

$$x^2 - 4x + 7$$

$$y'' - 4y' + 7y = 0$$

$$m^2 - 4m + 7 = 0$$

$$m^2 - 4m + 2^2 - 4 + 7$$

$$= (m-2)^2 + 3 = 0$$

$$\Rightarrow m-2 = \pm\sqrt{-3}$$

$$m = 2 \pm i\sqrt{3}$$

This means $e^{(2+i\sqrt{3})x}, e^{(2-i\sqrt{3})x}$ is F.S. of solns

$$= e^{2x} e^{i\sqrt{3}x}, e^{2x} e^{-i\sqrt{3}x}$$

General sol'n :

$$\begin{aligned}
 & c_1 e^{2x} e^{\sqrt{3}ix} + c_2 e^{2x} e^{-\sqrt{3}ix} \\
 & e^{2x} (e^{\sqrt{3}ix} + e^{-\sqrt{3}ix}) \\
 & = e^{2x} (\cos(\sqrt{3}x) + i\sin(\sqrt{3}x) + \cos(-\sqrt{3}x) + i\sin(-\sqrt{3}x)) \\
 & = e^{2x} (2\cos(\sqrt{3}x))
 \end{aligned}$$

Fact: If $m = a + bi$ is a root of C.P. then
 General sol'n is $e^{ax} (c_1 \cos(bx) + c_2 \sin(bx))$

That's basically 4.3.

4.4: Handling non-homogeneous 4.3 equations.

Solve the given differential equation by undetermined coefficients.

1. $y'' - 8y' + 20y = 200x^2 - 65xe^x$

C.P. : $m^2 - 8m + 20 = m^2 - 8m + 4^2 - 16 + 20$

$= (m-4)^2 + 4 \stackrel{\text{set}}{=} 0 \rightarrow$

$(m-4)^2 = -4$

$m-4 = \pm 2i$

$m = 4 \pm 2i$

Solns of homogeneous eqn: $e^{4x}(c_1 \cos(2x) + c_2 \sin(2x))$

Particular Sol'n

$y_p = Ax^2 + Bx + C + e^x(Ex + F)$

"D" is reserved operator in Maple.

Sub y_p into the ODE:

$2A + 13e^x Ex + 13e^x F - 6e^x E - 16Ax - 8B + 20Ax^2 + 20Bx + 20C$

Equate this w/ RHS $200x^2 - 65xe^x = \text{rhs}(x)$

I just wrote

LHS - RHS = 0 & collected terms. This gives.

$2A + 13e^x Ex + 13e^x F - 6e^x E - 16Ax - 8B + 20Ax^2 + 20Bx + 20C - 200x^2 + 65xe^x = 0$

This can be solved by back-sub,
or...

eqn1 := $2A - 8B + 20C = 0$

eqn2 := $13E + 65 = 0$

eqn3 := $13F - 6E = 0$

eqn4 := $-16A + 20B = 0$

eqn5 := $20A - 200 = 0$

$2A - 8B + 20C$

$= 0$

$13E + 65 = 0$

$-6E + 13F$

$-16A + 20B$

$= 0$

$20A$

$-200 = 0$

$2A - 8B + 20C$

$= 0$

$13E + 65 = 0$

$-6E + 13F = 0$

$-16A + 20B$

$= 0$

$20A$

$-200 = 0$

$$\begin{bmatrix} 2 & -8 & 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 13 & 0 & -65 \\ -16 & 20 & 0 & 0 & 0 & 0 \\ 20 & 0 & 0 & 0 & 0 & 200 \\ 0 & 0 & 0 & -6 & 13 & 0 \end{bmatrix}$$

Swap Row 3 ↑

$mymatrix := \langle 2, -8, 20, 0, 0, 0, 0; 0, 0, 0, 13, 0, -65; -16, 20, 0, 0, 0, 0; 0, 0, 0, -6, 13, 0; 20, 0, 0, 0, 0, 200 \rangle$

$$mymatrix := \begin{bmatrix} 2 & -8 & 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 13 & 0 & -65 \\ -16 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & 13 & 0 \\ 20 & 0 & 0 & 0 & 0 & 200 \end{bmatrix}$$

$ReducedRowEchelonForm(mymatrix)$

$$\begin{aligned} A &= W \\ B &= 0 \\ C &= \frac{11}{5} \\ E &= -5 \\ F &= -\frac{30}{13} \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 0 & \frac{11}{5} \\ 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & -\frac{30}{13} \end{bmatrix}$$

11

CHAPTER 3 WRITTEN :

$$\frac{dT}{dt} = k(T^4 - T_m^4)$$

$$\frac{dT}{T^4 - T_m^4}$$

$$\frac{1}{T^4 - T_m^4}$$

use Wolfram Alpha
I tried doing Partial
fractions.

$$\begin{aligned} \frac{dT}{dt} &= k(T^4 - T_m^4) \\ &= k\left((T_m + T - T_m)^4 - T_m^4\right) \\ &= k\left(\left(T_m\left(1 + \frac{T - T_m}{T_m}\right)\right)^4 - T_m^4\right) \\ &= k\left(T_m^4\left(1 + \frac{T - T_m}{T_m}\right)^4 - T_m^4\right) \\ &= kT_m^4\left(\left(1 + \frac{T - T_m}{T_m}\right)^4 - 1\right) \end{aligned}$$

$$\begin{array}{ccccccc} & & 1 & & & & \\ & 1 & & 1 & & & \\ & & 2 & & 1 & & \\ 1 & 3 & & 3 & & 1 & \\ 1 & 4 & 6 & 4 & 1 & & \end{array}$$

Binomial Theorem Review

$$(a+b)^4 = \sum_{k=0}^4 \binom{4}{k} a^{4-k} b^k = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \quad n \text{ is a positive integer}$$

$\binom{n}{k} = nC_k =$ "n choose k" = # of ways you can choose k things from n things. = the # of subsets of a set of n elements that contain k elements.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\dots(n-k+1)\cancel{(n-k)}\dots\cancel{(3)}\cancel{(2)}\cancel{(1)}}{(k(k-1)\dots(3)(2)(1))\cancel{(n-k)}\cancel{(n-k-1)}\dots\cancel{(n-k+2)}\cancel{(2)}\cancel{(1)}}$$

$$\binom{5}{3} = \frac{5!}{3!2!} = \frac{5 \cdot 4}{2!} = 10$$

$$\binom{4}{2} = \frac{4 \cdot 3}{2} = 6$$

$\{a, b, c, d\}$

$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$

$$\{a, b\} = \{b, a\}$$

$$k T_m^4 \left(\left(1 + \frac{T - T_m}{T_m} \right)^4 - 1 \right)$$

$$(1)^4 + 4(1)^3 \left(\frac{T - T_m}{T_m} \right) + 6(1)^2 \left(\frac{T - T_m}{T_m} \right)^2 + 4(1) \left(\frac{T - T_m}{T_m} \right)^3 + \left(\frac{T - T_m}{T_m} \right)^4$$

$$= 1 + 4 \left(\frac{T - T_m}{T_m} \right) + 6 \left(\frac{T - T_m}{T_m} \right)^2 + 4 \left(\frac{T - T_m}{T_m} \right)^3 + \left(\frac{T - T_m}{T_m} \right)^4$$

$T - T_m$ "small"
 $\rightarrow$ 1st 3 terms
 Throw out the $\left(\frac{T - T_m}{T_m} \right)^2$ term small

$$\frac{dT}{dt} \approx k T_m^4 \left(1 + 4 \frac{T - T_m}{T_m} - 1 \right) = 4 k T_m^4 \left(\frac{T - T_m}{T_m} \right) = 4 k T_m^3 (T - T_m)$$

So it IS Newton's Law
with ITS constant $K = 4 k T_m^3$