The indicated function $y_1(x)$ is a solution of the following differential equation.

1. $y'' + 2y' + y = 0; \quad y_1 = xe^{-x}$

Use reduction of order, or the formula

$$y_2 = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx$$

as instructed, to find a second solution $y_2(x)$.

We find y_2 using the theory of 4.2.

The formulaic approach is probably quicker, but there are better ways to solve these homogeneous linear ODEs with constant coefficients in Section 4.3.

Assume $y_2 = u(x)\,y_1(x)$

$y_2' = u'y_1 + uy_1'$

$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$

$y_2'' = u''y_1 + 2u'y_1' + uy_1''$

$y_2'' + 2y_2' + y_2 = 0;$

$\cancel{u''y_1} + \cancel{2u'y_1'} + \cancel{uy_1''} + 2\cancel{(u'y_1 + uy_1')} + \cancel{uy_1} = 0$

$2u'y_1 + \cancel{2uy_1'}$

$u''y_1 + 2u'y_1' + u\underbrace{(y_1'' + 2y_1' + y_1)}_{0} + 2u'y_1 = 0$

$u''y_1 + 2u'y_1' + 2u'y_1 = 0$

$w = u' \longrightarrow$

$w'y_1 + 2wy_1' + 2wy_1 = 0$

$y_1 = xe^{-x} \Longrightarrow y_1' = e^{-x} - xe^{-x}$

$xe^{-x}w' + 2(e^{-x} - \underline{xe^{-x}})w + \underline{2xe^{-x}w} = 0$

$xe^{-x}w' + 2e^{-x}w = 0$

$w' + \frac{2}{x}w = 0 \quad$ Stay away from $x = 0$

$e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln|x| + c} = Ke^{\ln(x^2)} = Kx^2 = x^2$ if $K = 1$.

$x^2 w' + x^2\left(\frac{2}{x}\right)w = 0$

$x^2 w' + 2xw = 0$

$\frac{d}{dx}\left[x^2 w\right] = 0$

$x^2 w = c$

$w = \frac{c}{x^2} = u' \longrightarrow$

$u = \int \frac{c}{x^2}dx = -\frac{c}{x} + \hat{c} \quad$ Let $\hat{c} = 0$, Let $c = 1$

$y_2 = uy_1(x) = \frac{1}{x}(xe^{-x}) = \boxed{e^{-x} = y_2}$

The indicated function $y_1(x)$ is a solution of the following differential equation.

4. $4x^2y'' + y = 0; \quad y_1 = x^{1/2}\ln(x)$

Use reduction of order, or the formula

$$y_2 = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx$$

as instructed, to find a second solution $y_2(x)$.

Reduction of order for 2nd ORDER Homogeneous ODE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

Divide thru by $a_2(x)$:

$$y'' + P(x)y' + Q(x)y = 0$$

Given sol'n y_1, assume y_2 is an (the) other sol'n of the form $y_2(x) = u(x)y_1(x)$ for some dif'b'l $u(x)$.

Then $y_2' = u'y_1 + uy_1'$

& $y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$

$= u''y_1 + 2u'y_1' + uy_1''$

Substitute y_2 into the eq'n

$$u''y_1 + 2u'y_1' + uy_1'' + P(x)(u'y_1 + uy_1') + Q(x)(uy_1) = 0$$

$$u''y_1 + 2u'y_1' + uy_1'' + u'P(x)y_1 + uP(x)y_1' + uQ(x)y_1$$

$$= u''y_1 + 2u'y_1' + u'P(x)y_1 + uy_1'' + uP(x)y_1' + uQ(x)y_1$$

$$= u''y_1 + 2u'y_1' + u'P(x)y_1 + u\underbrace{(y_1'' + P(x)y_1' + Q(x)y)}_{} =$$

$$= u''y_1 + 2u'y_1' + u'P(x)y_1 = 0 \qquad y_1 \text{ is a sol'n.}$$

$$= y_1u'' + 2y_1'u' + P(x)y_1u'$$

$$\Rightarrow u'' + \frac{2y_1'u'}{y_1} + \frac{P(x)y_1u'}{y_1}$$

$$= u'' + \left(\frac{2y_1'}{y_1} + P(x)\right)u' = 0$$

$\swarrow$ $\longrightarrow$ Intuition

ROF

$$w' + \left(\frac{2y_1'}{y_1} + P(x)\right)w = 0$$

$$w' = \left(-2\frac{y_1'}{y_1} - P(x)\right)w$$

$$\frac{w'}{w} = -2\frac{y_1'}{y_1} - P(x)$$

$$\ln|w| = -2\ln|y_1| - \int P(x)\,dx + C_1$$

$$\ln|w| = -\ln(y_1^2) - \int P(x)\,dx + C_1$$

Eq'n is separable. We could just carry on with

$$u'' + \left(\frac{2y_1'}{y_1} + P(x)\right)u' = 0$$

$$\frac{u''}{u'} = -2\frac{y_1'}{y_1} - P(x)$$

$\ln|u'| =$ etc.

$$u' = w = e^{-\ln(y_1^2) - \int P(x)\,dx + C_1} = K\frac{1}{y_1^2}e^{-\int P(x)\,dx}, \text{ where } K = e^{C_1}$$

$$\Rightarrow u = K\int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx + C_2$$

. Let $C_2 = 0$, $K = 1$ $\longrightarrow$

$$u = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$\Rightarrow y_2(x) = u(x)\,y_1(x) = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1(x)^2} \quad \blacksquare$$

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n$$

When *k* is a positive integer, we have the special case everyone learns in college algebra. The series terminates after *k* + 1 terms (accounting for the 0th term that's constant).

§4.2 #7

$u' = -\frac{x}{4}e^{-x} - \frac{1}{16}e^{-x} + c_1 e^{3x}$

$\rightarrow u = \int\left(-\frac{x}{4}e^{-x} - \frac{1}{16}e^{-x} + c_1 e^{3x}\right)dx$

$= -\frac{1}{4}\int xe^{-x}\,dx - \frac{1}{16}\int e^{-x}\,dx + c_1\int e^{3x}\,dx$

$= -\frac{1}{4}\left(uv - \int v\,du\right) + \frac{1}{16}e^{-x} + c_1\left(\frac{1}{3}\right)e^{3x} + c_2$

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$= -\frac{1}{4}x(-e^{-x}) - \frac{1}{4}\left(-\int -e^{-x}\,dx\right) + \frac{1}{16}e^{-x} + \frac{1}{3}c_1 e^{3x} + c_2$

$u(x) = \frac{1}{4}xe^{-x} + \frac{1}{4}e^{-x} + \frac{1}{16}e^{-x} + \frac{1}{3}c_1 e^{3x}$ Let $c_2 = 0$, $c_1 = 3$

$u(x)y_1(x) = \frac{1}{4}x + \frac{1}{4} + \frac{1}{16} + e^{4x} = \underbrace{\frac{1}{4}x + \frac{5}{16}}_{y_p\text{ - DELETE}} + e^{4x} = y_2$ is same y_2 Ajay got.

Given $y_1(x) = e^x$

$y'' - 5y' + 4y = x$

$y'' - 5y' + 4y = 0$

$y_p = Ax^2 + Bx + C$

or $y_p = Ax + B$

Try that

$y_p = Ax + B$
$y_p' = A$
$y_p'' = 0$

Sub IN:

$0 + -5A + 4(Ax + B) = x$

$-5A + 4B = 0$

$4A = 1$

$A = \frac{1}{4} \Rightarrow$

$-5(\frac{1}{4}) + 4B = 0$

$-\frac{5}{4} + 4B = 0$

$4B = \frac{5}{4}$

$B = \frac{5}{16}$

$$\boxed{y_p = \frac{1}{4}x + \frac{5}{16}}$$

$$T^4 - T_m^4 = (T_m + T - T_m)^4 - T_m^4$$

$$= T_m^4 + +4T_m^3(T-T_m) + 6T_m^2(T-T_m)^2 + 4T_m(T-T_m)^3 + (T-T_m)^4 - T_m^4$$

$$(T_m + (T-T_m))^4 - T_m^4$$

$$= \left(T_m\left(1 + \frac{(T-T_m)}{T_m}\right)\right)^4 - T_m^4$$

$$\frac{4\cdot 3}{2} = 6$$

$$= T_m^4\left(1+\frac{T-T_m}{T_m}\right)^4 - T_m^4$$

$$= T_m^4\left[\left(1+\left(\frac{T-T_m}{T_m}\right)\right)^4 - 1\right]$$

$$= T_m^4\left[\sum_{n=0}^{\infty} \binom{k}{n}\left(\frac{T-T_m}{T_m}\right)^n - 1\right]$$

$$= T_m^4\left[1 + 4\left(\frac{T-T_m}{T_m}\right) + \binom{4}{2}\left(\frac{T-T_m}{T_m}\right)^2 + \binom{4}{3}\left(\frac{T-T_m}{T_m}\right)^3 + \left(\frac{T-T_m}{T_m}\right)^4 - 1\right]$$

$$= T_m^4\left[4\left(\frac{T-T_m}{T_m}\right) + 6\left(\frac{T-T_m}{T_m}\right)^2\right]$$