Quick intro to some linear algebra

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a 2x2 array, or MATRIX of real numbers

Sometimes we write $[a_{ij}]$, where the a_{ij} represents the entry in the i^{th} row and j^{th} column.

For a 2x1 matrix or VECTOR $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, define the product

$A\bar{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \bar{b}$. We write $\bar{b} = [b_{ij}] = \left[\sum_{k=1}^{2} a_{ik}x_k \right]$

EXAMPLE Consider the system of linear equations

E1 $c_1 + c_2 = 1$
E2 $4c_1 + c_2 = 3$

Define $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Then $A\bar{x} = \bar{b}$

GAUSSIAN ELIMINATION:

$-4E1$: $-4c_1 - 4c_2 = -4$

$E2$: $4c_1 + c_2 = 3$

$-4E1 + E2$: $-3c_2 = -1$

$\boxed{c_2 = \frac{1}{3}}$

BACK-SUB:

$c_1 + c_2 = c_1 + \frac{1}{3} = 1 \longrightarrow \boxed{c_1 = \frac{2}{3}}$

Gauss with Matrix:

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & 1 & 3 \end{array}\right] \begin{array}{l} R1 \\ -4R1+R2 \end{array} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -3 & -1 \end{array}\right]$$

$$\begin{array}{l} R1 \\ -\frac{1}{3}R2 \end{array} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{3} \end{array}\right]$$

This says $c_2 = \frac{1}{3}$ & $c_1 + c_2 = 1$

BACK-SUB

GAUSS-JORDAN:

Instead of back-substitution, we can elminate the '1' in the 1,2 spot with another elimination.

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{3} \end{array}\right] \begin{array}{l} -R2+R1 \\ R2 \end{array} \left[\begin{array}{cc|c} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{1}{3} \end{array}\right]$$

This says $c_1 = \frac{2}{3}$, $c_2 = \frac{1}{3}$

we can extend $\underset{2\times2}{A}\ \underset{2\times1}{\bar{x}} = \underset{2\times1}{\bar{b}}$ to $\underset{2\times2}{A} \cdot \underset{2\times2}{B} = \underset{2\times2}{C}$, by replicating the procedure of Matrix times vector for each column of B

$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -6 \\ 3 & 2 \end{bmatrix}$ Think of B as 2 vectors, side by side then $AB = A[\bar{b}_1\ \ \bar{b}_2] = [A\bar{b}_1\ \ A\bar{b}_2]$

$$AB = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & -6 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1(1)+(1)(3) & 1(-6)+1(2) \\ 4(1)+(1)(3) & 4(-6)+1(2) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 \\ 7 & -22 \end{bmatrix}$$

MATRIX INVERSE:

Define $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$ **multiplicative identity in the ring of 2x2 matrixes**

$AI = IA = A$

The inverse of a 2x2 matrix A is denoted by A^{-1} and has the property $AA^{-1} = A^{-1}A = I$

To $\underline{\text{FIND}}$ A^{-1}, we solve $A\bar{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A\bar{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and make the matrix $A^{-1} = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ $\left(\bar{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} \quad \bar{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} \right)$

We can solve for BOTH columns of A^{-1}, by writing $[A \mid I]$ and doing GAUSS-JORDAN:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -3 & -4 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{3} & -\frac{1}{3} \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{4}{3} & -\frac{1}{3} \end{array}\right]$$

Here's the scheme:

$$[A \mid I] \underbrace{\sim \cdots \sim}_{\substack{\text{GAUSS-JORDAN} \\ \text{ROW-REDUCTION} \\ \text{ON } [A \mid I]}} [I \mid A^{-1}]$$

By this, we know that

$$\begin{bmatrix}1&1\\4&1\end{bmatrix}\begin{bmatrix}-\frac{1}{3}&\frac{1}{3}\\ \frac{4}{3}&-\frac{1}{3}\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}.$$ check: $\begin{bmatrix}-\frac{1}{3}+\frac{4}{3}&-\frac{1}{3}+\frac{1}{3}\\ -\frac{4}{3}+\frac{4}{3}&\frac{4}{3}-\frac{1}{3}\end{bmatrix}$ ✓

Apply to previous:

Want $\bar{x} \ni A\bar{x}=\bar{b} \Longrightarrow$

$$A^{-1}(A\bar{x}) = A^{-1}\bar{b} = \frac{1}{3}\begin{bmatrix}-1&1\\4&-1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}-1+3\\4-3\end{bmatrix}=\frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix}=\boxed{\begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\end{bmatrix}=\bar{x}}$$

DETERMINANT of A is

$a_{11}a_{22}-a_{12}a_{21}$ $\qquad |A| = \det(A) = \begin{vmatrix}1&1\\4&1\end{vmatrix} = 1-4=-3$

$$\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}$$

Cramer's Method for 2x2:

$$A^{-1}=\frac{1}{|A|}\begin{bmatrix}a_{22}&-a_{12}\\-a_{21}&a_{11}\end{bmatrix} 4, \quad A=\begin{bmatrix}1&1\\4&1\end{bmatrix}\Longrightarrow A^{-1}=\frac{1}{1-4}\begin{bmatrix}1&-1\\-4&1\end{bmatrix}$$

check: $-\frac{1}{3}\begin{bmatrix}1&-1\\-4&1\end{bmatrix}\begin{bmatrix}1&1\\4&1\end{bmatrix}=-\frac{1}{3}\begin{bmatrix}-3&0\\0&-3\end{bmatrix}=I$ ✓

$$A=\begin{bmatrix}1&2\\3&4\end{bmatrix}\Longrightarrow A^{-1}=\frac{1}{-2}\begin{bmatrix}4&-2\\-3&1\end{bmatrix}$$

check $A^{-1}A=-\frac{1}{2}\begin{bmatrix}4&-2\\-3&1\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}-2&0\\0&-2\end{bmatrix}=I$ ✓

To understand $|A|$ for larger matrices, start with 1x1

$A=[5]\Longrightarrow |A|=5$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |A| = (-1)^{1+1} a_{11} |a_{22}| + (-1)^{1+2} a_{12} |a_{21}| = a_{11}a_{22} - a_{12}a_{21}$$

$[a_{22}]$ = matrix obtained by deletion row 1, column 1.

= expansion by minors, across row 1

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = (-1)^{1+1} a_{11} A_{11} + (-1)^{1+2} a_{12} A_{12} + (-1)^{1+3} a_{13} A_{13},$$

$$= a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$$

where A_{12} = **Determinant of the submatrix obtained by deleting Row 1 and Row 2 from A.**

This rule applies to 2x2 :

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (-1)^{1+1} a_{11} A_{11} + (-1)^{1+2} a_{12} A_{12}$$

$$= a_{11} |a_{22}| - a_{12} |a_{21}|$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

When you see it this way, you can write Cramer's Rule for a 2x2 as follows:

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} (-1)^{1+1}A_{11} & (-1)^{1+2}A_{21} \\ (-1)^{2+1}A_{12} & (-1)^{2+2}A_{22} \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Notice the subscripts of the A's are reversed.

Looked at in this way, which we won't, much, The inverse of a 3x3 by Cramer's Rule is

$$\frac{1}{|A|}\begin{bmatrix} A_{11} & -A_{21} & A_{31} \\ -A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Now, we won't be doing inverses this way, in practice, BUT we do need to know how to do determinants of 3x3, 4x4 matrices.

$$A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \end{array} \Rightarrow |A| = 1(5(10)-8(6)) - 2(4(10)-6(7))$$

$$+3(4(8)-5(7))$$

$$= 1(2)-2(-2)+3(-3)$$

$$= 2+4-9 = -3$$

we can take determinants of Matrices w/ function entries.

$$\begin{vmatrix} e^t & \sin(t) \\ e^t & \cos(t) \end{vmatrix} = e^t\cos(t) - e^t\sin(t)$$

LINEAR INDEPENDENCE

we say that n vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$ or n functions $f_1(x), \ldots, f_n(x)$ are linearly independent if the only solution to

the equation $c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_n\bar{v}_n = 0$ is $c_1 = c_2 = \cdots = c_n = 0$

and the eq'n $k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x) = 0$ for all x has the unique (trivial) solution $k_1 = \cdots = k_n = 0$