Linear ODEs

4.1.1 Initial-Value and Boundary-Value Problems

Initial-Value Problem In Section 1.2 we defined an initial-value problem for a general nth-order differential equation. For a linear differential equation an **nth-order initial-value problem (IVP)** is

$$\textbf{Solve:} \quad a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

$$\textbf{Subject to:} \quad y(x_0) = y_0, \quad y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}.$$

Theorem 4.1.1 Existence of a Unique Solution

Let $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem (1) exists on the interval and is unique.

E1 $3y''' + 5y'' - y' + 7y = 0, \quad y(1) = 0, \quad y'(1) = 0, \quad y''(1) = 0$

Note $y=0$ satisfies eq'n & IC's $\Rightarrow$ $y=0$ is ! sol'n.

E2 $y'' - 4y = 12x, \quad y(0) = 4, \quad y'(0) = 1.$ **has a ! solution.**

Boundary-Value Problem Another type of problem consists of solving a linear differential equation of order two or greater in which the dependent variable y or its derivatives are specified at *different points*. A problem such as

$$\textbf{Solve:} \quad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$\textbf{Subject to:} \quad y(a) = y_0, \quad y(b) = y_1$$

Other BCs that are possible. Main thing is for 2nd-order, you're going to need 2 BCs.

$$\begin{aligned} y'(a) &= y_0, & y(b) &= y_1 \\ y(a) &= y_0, & y'(b) &= y_1 \\ y'(a) &= y_0, & y'(b) &= y_1, \end{aligned}$$

All of these would fall under the generic BCs:

$$\begin{aligned} \alpha_1 y(a) + \beta_1 y'(a) &= \gamma_1 \\ \alpha_2 y(b) + \beta_2 y'(b) &= \gamma_2. \end{aligned}$$

Existence and uniqueness of soluutions for BVPs doesn't follow Theorem 4.1.1.

There may be a ! soln, many solutions, or no solutions, even if continuity conditions on the coefficient are in force and the leading coefficient is never zero on the interval in question.

E3 In Example 9 of Section 1.1 we saw that the two-parameter family of solutions of the differential equation $x'' + 16x = 0$ is

$$x = c_1 \cos 4t + c_2 \sin 4t. \tag{2}$$

If it were an IVP, there would be a ! soln. But BC's: $x(0) = 0 = x(\frac{\pi}{2}) \Longrightarrow$ many sol'ns.

$x(0) = c_1 = 0 \Rightarrow \boxed{c_1 = 0}$

$x(\frac{\pi}{2}) = c_1 = 0 \rightarrow \boxed{c_1 = 0}$ again! No restriction on c_2,

... sin(4t) is sol'n $\forall c \in \mathbb{R}$

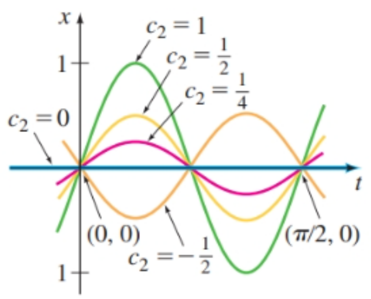


Figure 4.1.2 Solution curves for BVP in part (a) of Example 3

Homogeneous Equations

A linear nth-order differential equation of the form

$$\sum_{k=0}^{n} a_k \frac{d^k y}{dx^k} = a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (6)$$

or $\sum_{k=0}^{n} a_{n-k} \frac{d^{n-k}}{dx^{n-k}}$ (Descending order, as original)

is said to be homogeneous.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad (7)$$

is said to be nonhomogeneous.

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y =$

Differential Operators: $\frac{d^n}{dx^n} = D^n$

$$\sum_{k=0}^{n} a_{n-k} \frac{d^{n-k}}{dx^{n-k}}(y) = \sum_{k=0}^{n} a_{n-k} D^{n-k} y = \left(\sum_{k=0}^{n} a_{n-k} D^{n-k}\right) y,$$ because

differentiation respects addition and scalar multiplication.

This is an example of a LINEAR OPERATOR:

L is a linear operator means

$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$ $\quad$ α, β constant, f, g functions

Differentiation:

$$\frac{d}{dx}\left[6x^2 + 5\sin(x)\right] = 6\frac{d}{dx}(x^2) + 5\frac{d}{dx}(\sin(x))$$

$$= 12x + 5\cos(x)$$

$$D(af + bg) = aDf + bDg$$

Superposition Principle: (Old sedimentary rocks lie under young rocks.)

Theorem 4.1.2 Superposition Principle—Homogeneous Equations

Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous nth-order differential equation (6) on an interval I. Then the linear combination

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x),$$

where the c_i, $i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution on the interval.

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x), = \sum_{i=1}^{k} c_i y_i(x)$$

is a linear combination on the functions $y_1, y_2, \ldots, y_k$

Proof Let $Ly = \sum_{i=0}^{n} a_i D^i y = 0$ and $y_1, y_2, \ldots, y_k$ be sol'ns.

Then $L\left(\sum_{j=1}^{k} c_j y_j(x)\right) = \sum_{i=0}^{n} a_i D^i\left(\sum_{j=1}^{k} c_j y_j(x)\right) = \sum_{i=0}^{n}\left(\sum_{j=1}^{k} c_j\left(a_i D^i y_j(x)\right)\right)$

$$= \sum_{i=0}^{n}\left(c_1 a_i D^i y_1(x) + c_2 a_i D^i y_2(x) + \cdots + c_k a_i D^i y_k(x)\right)$$

$$\begin{aligned}
= {} & c_1 a_0 y_1(x) + c_2 a_0 y_2(x) + \cdots + c_k a_0 y_k(x) \\
+ {} & c_1 a_1 D y_1 + c_2 a_1 D y_2 + \cdots + c_k a_1 D y_k \\
+ {} & c_1 a_2 D^2 y_1 + c_2 a_2 D^2 y_2 + \cdots + c_k a_2 D^2 y_k \\
+ {} & \cdots \\
+ {} & c_1 a_n D^n y_1 + c_2 a_n D^n y_2 + \cdots + c_k a_n D^n y_k
\end{aligned}$$

$$c_1 \sum_{i=0}^{n} a_i D^i y_1 + c_2 \sum_{i=0}^{n} a_i D^i y_2 + \cdots + c_k \sum_{i=0}^{n} a_i D^i y_k$$

$$\overset{\shortparallel}{c_1(0)} + c_2(0) + \cdots + c_k(0) = 0 \Rightarrow$$

$\sum_{j=1}^{k} c_j y_j$ is a sol'n, too!

Bottom line: linear combinations of solutions to a linear homogeneous ODE are also solutions.

Corollaries to Theorem 4.1.2

(A) A constant multiple $y = c_1y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear differential equation is also a solution.

(B) A homogeneous linear differential equation always possesses the trivial solution $y = 0$.

D 4.1.1

LINEAR INDEPENDENCE

we say that (n vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$) or n functions $f_1(x), \ldots, f_n(x)$

are linearly independent if the only solution to

(the equation $c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_n\bar{v}_n = 0$ is $c_1 = c_2 = \cdots = c_n = 0$)

and the eq'n $k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x) = 0$ for all x has

the unique (trivial) solution $k_1 = \cdots = k_n = 0$

For now, in *this* course, we're more interested in the linear independence of *functions*.

Linear independence means that only trivial linear combinations of the set sums to zero.

E5 The set of functions $f_1(x) = \cos^2 x$, $f_2(x) = \sin^2 x$, $f_3(x) = \sec^2 x$, $f_4(x) = \tan^2 x$ is linearly dependent on the interval $(-\pi/2, \pi/2)$ because

$$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0$$

$$c_1(1 - \sin^2(x)) + c_2 \sin^2(x) + c_3 \cdot (\tan^2(x) + 1) + c_4 \tan^2(x)$$

$$= c_1 - c_1 \sin^2(x) + c_2 \sin^2(x) + c_3 \tan^2(x) + c_3 + c_4 \tan^2(x)$$

Let $\boxed{c_1 = -c_3}$

$$= -c_3 + c_3 \sin^2(x) + c_2 \sin^2(x) + c_3 \tan^2(x) + c_4 \tan^2(x) + c_3$$

$$= c_3 \sin^2(x) + c_2 \sin^2(x) + c_3 \tan^2(x) + c_4 \tan^2(x)$$

Let $\boxed{c_3 = -c_2}$

$$= -c_2 \tan^2(x) + c_4 \tan^2(x)$$

Let $\boxed{c_4 = c_2}$

Then we have proved it's dependent.

Let $c_2 = 1$, $c_3 = -1$, $c_1 = 1$, $c_4 = 1$

Then

$$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0$$

$$\cos^2(x) + \sin^2(x) - \sec^2(x) + \tan^2(x)$$

$$= 1 - \sin^2(x) + \sin^2(x) - \tan^2(x) - 1 + \tan^2(x) = 0 \quad \checkmark$$

There's a quicker way to establish linear independence of a set of functions, called the Wronskian.

In the homework, they will ask if they're independent and if they're not, it will want you to find a solution for the constants that make the linear combination = 0. But if they ARE independent, you're going to be done.

Definition 4.1.2 Wronskian

Suppose each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where the primes denote derivatives, is called the **Wronskian** of the functions.

Trouble is, you may not remember how to do determinants!

I will compute a few, and explain what I'm doing, but I don't think we have time in class to develop the theory very much. So I created a quick intro to linear algebra sufficient to work the exercises.

Click Here for Brief Linear Algebra Intro

$A = \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}, \quad \bar{x} = \underset{2\times 1}{\begin{bmatrix} 5 \\ 6 \end{bmatrix}}$

$A\bar{x} = \begin{bmatrix} 1(5)+2(6) \\ 3(5)+4(6) \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix} = .$

$c_1 - 5c_2 = 7$

$c_1 + 4c_2 = -1$

$A = \begin{bmatrix} 1 & -5 \\ 1 & 4 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$ and

we want $\bar{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ so that

$A\bar{x} = \bar{b}$

$A\bar{x} = \begin{bmatrix} 1 & -5 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$

$\begin{bmatrix} c_1 - 5c_1 \\ c + 4c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$

Cool thing is if we had an inverse for A, say A^{-1}

If $A^{-1}A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = multiplicative identity

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$

Extend this to Matrices:

$\underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}\underset{2\times 2}{\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}} = \begin{bmatrix} (1)(5)+2(7) & 1(6)+2(8) \\ 3(5)+4(7) & 3(6)+4(8) \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}, \quad A^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ -4 & 1 \end{bmatrix}$

$A\bar{x} = \bar{b}$

$A^{-1}A\bar{x} = A^{-1}\bar{b}$

$I\bar{x} = A^{-1}\bar{b}$

$\bar{x} = A^{-1}\bar{b}$

Theorem 4.1.3 Criterion for Linearly Independent Solutions

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear nth-order differential equation (6) on an interval I. Then the set of solutions is **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

Fact: If you have *n* solutions to an *n*th-order homogeneous ODE and they form a linearly independent set, then you know you have captured *all* the solutions!

Fact: You can sometimes make a set of functions linearly independent by restricting the interval of definition.

2x2 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3x3 EXPAND by minors across row 1:

$$\begin{vmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - hf) - b(di - gf) + c(dh - ge)$$

Wronskian for the set $e^t, \sin(t)$

$$\begin{vmatrix} e^t & \sin(t) \\ e^t & \cos(t) \end{vmatrix} = e^t\cos(t) - e^t\sin(t) = 0 \text{ when } t = \frac{\pi}{4}$$

(or anywhere $\tan(t) = 1$)

$e^t(\cos(t) - \sin(t)) = 0 \Rightarrow$

$\cos(t) = \sin(t)$

$\tan(t) = 1$

You'd have to restrict away from $t = \frac{\pi}{4}$

Example 5

$with(VectorCalculus):$

$Wronskian([\cos(x)^2, \sin(x)^2, \sec(x)^2, \tan(x)^2], x, 'determinant')$

$$\left[\begin{array}{cccc} \cos(x)^2 & \sin(x)^2 & \sec(x)^2 & \tan(x)^2 \\ -2\cos(x)\sin(x) & 2\cos(x)\sin(x) & 2\sec(x)^2\tan(x) & 2\tan(x)\left(1+\tan(x)^2\right) \\ 2\sin(x)^2-2\cos(x)^2 & -2\sin(x)^2+2\cos(x)^2 & 4\sec(x)^2\tan(x)^2+2\sec(x)^2\left(1+\tan(x)^2\right) & 2\left(1+\tan(x)^2\right)^2+4\tan(x)^2\left(1+\tan(x)^2\right) \\ 8\cos(x)\sin(x) & -8\cos(x)\sin(x) & 8\sec(x)^2\tan(x)^3+16\sec(x)^2\tan(x)\left(1+\tan(x)^2\right) & 16\left(1+\tan(x)^2\right)^2\tan(x)+8\tan(x)^3\left(1+\tan(x)^2\right) \end{array}\right], 0$$

$|w| = 0 \Rightarrow$ Linearly Dependent

Pg 125: "It follows... that when you have *n* solutions of the homogeneous linear *n*th-order ODE given in (6) on an interval *I*, then the Wronskian is either identically zero or never zero on the interval.

I have to think about this a little bit. Still not quite seeing it, although if it's true, it gives us an easy way to determine independence/dependence. Just substitute one value of *t* . If you get 0, then they're dependent. If you get something nonzero, then they're independent?!

I guess it just depends on the interval of definition. More on this, later. I don't quite remember all of the theory.

In any case, we can clobber Wronskians of any size with Maple and solve messy equations with Maple...

Right-click here to download (Save as...) Maple that accompanies this section.

For 2x2 and 3x3, these determinants aren't all that technically difficult.

Click here to view PDF of the Maple

Definition 4.1.3 Fundamental Set of Solutions

Any set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of the homogeneous linear nth-order differential equation (6) on an interval I is said to be a **fundamental set of solutions** on the interval.

Theorem 4.1.4 Existence of a Fundamental Set

There exists a fundamental set of solutions for the homogeneous linear nth-order differential equation (6) on an interval I.

Theorem 4.1.5 General Solution—Homogeneous Equations

Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous linear nth-order differential equation (6) on an interval I. Then the **general solution** of the equation on the interval is

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x),$$

where c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

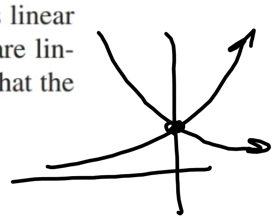
$c_1e^{3x}+c_2e^{-3x}=0$

$c_1e^0+c_2e^0=c_1+c_2=0$

$\Rightarrow c_1=-c_2$

E7 The functions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of the homogeneous linear equation $y'' - 9y = 0$ on the interval $(-\infty, \infty)$. By inspection the solutions are linearly independent on the x-axis. This fact can be corroborated by observing that the Wronskian

$(0,\infty)$ or $(-\infty,0)$?



$$W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$$

for every x. We conclude that y_1 and y_2 form a fundamental set of solutions, and consequently, $y = c_1e^{3x} + c_2e^{-3x}$ is the general solution of the equation on the interval. ■

Clobbering it with Maple:

$Wronskian([\exp(3\cdot x), \exp(-3\cdot x)], x, 'determinant')$

Unevaluated determinant **Evaluated determinant**

$$\begin{bmatrix} e^{3x} & e^{-3x} \\ 3\,e^{3x} & -3\,e^{-3x} \end{bmatrix}, -6\,e^{3x}\,e^{-3x}$$

$simplify(-6\,e^{3x}\,e^{-3x})$

Click here for PDF of this Maple session.

$$-6$$

By hand: $\begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = e^{3x}(-3e^{-3x}) - e^{-3x}(3e^{3x})$

$= -3-3 = -6 \neq 0 \;\forall\, x \in \mathbb{R} \longrightarrow$

Linearly independent

E8 - We just got done proving the exp(3x) and exp(-3x) are linearly independent. They are also solutions of the differential equation

$$y'' - 9y = 0$$

Thus $c_1 e^{3x} + c_2 e^{-3x}$ is the general solution.

But wait! $y = 4\sinh(3x) - 5e^{-3x}$ is ALSO a solution

$f := x \to 4 \cdot \sinh(3 \cdot x) - 5 \cdot \exp(-3 \cdot x)$

$$f := x \mapsto 4 \cdot \sinh(3 \cdot x) - 5 \cdot e^{-3 \cdot x}$$

$fp := D(f)$

$$fp := x \mapsto 12 \cdot \cosh(3 \cdot x) + 15 \cdot e^{-3 \cdot x}$$

$fpp := D(fp)$

$$fpp := x \mapsto 36 \cdot \sinh(3 \cdot x) - 45 \cdot e^{-3 \cdot x}$$

$fpp(x) - 9 \cdot f(x)$

$$0$$

Either theory is wrong, or $\exists\ c_1, c_2 \ni c_1 e^{3x} + c_2 e^{-3x} = 4\sinh(3x) - 5e^{-3x}$

The latter proves to be true, when we recall

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

Let's find c_1, c_2:

$$c_1 e^{3x} + c_2 e^{-3x} = 4\sinh(3x) - 5e^{-3x} = 3\left(\frac{e^{3x} - e^{-3x}}{2}\right) - 5e^{-3x}$$

$$= \frac{4}{2}e^{3x} - \frac{4}{2}e^{-3x} - 5e^{-3x}$$

$$= 2e^{3x} - 7e^{-3x}$$

$c_1 = 2,\ c_2 = -7$ does the trick!

Nonhomogeneous Solutions

Consider the nonhomogeneous version of (6), namely

(7) $a_n D^n y + a_{n-1} D^{n-1} y + \cdots + a_1 D y + a_0 y = g(x)$

We're going to build solutions to this new problem by solving the homogeneous equation (6) (g(x) = 0), and then finding a *particular* solution that yields g(x) on the right hand side.

The new solution will be the sum of the solution to (6) and the particular solution. The solution to the homogeneous equation y_c and the particular solution y_p

y_c = complementary solution = Kernel of the Diff. operator.

y_p = particular solution

The stuff that gets sent to 0.

By substitution the function $y_p = -\frac{11}{12} - \frac{1}{2}x$ is readily shown to be a particular solution of the nonhomogeneous equation

$$y''' - 6y'' + 11y' - 6y = 3x. \tag{11}$$

To write the general solution of (11), we must also be able to solve the associated homogeneous equation

$$y''' - 6y'' + 11y' - 6y = 0.$$

But in Example 9 we saw that the general solution of this latter equation on the interval $(-\infty, \infty)$ was $y_c = c_1e^x + c_2e^{2x} + c_3e^{3x}$. Hence the general solution of (11) on the interval is

$$y = y_c + y_p = \underbrace{c_1e^x + c_2e^{2x} + c_3e^{3x}}_{y_c} \underbrace{- \frac{11}{12} - \frac{1}{2}x}_{y_p}.$$

$y''' - 6y'' + 11y' - 6y = 0$

$(D^3 - 6D^2 + 11D - 6)y = 0 \longrightarrow D = 1, 2, 3$

$\longrightarrow y = c_1e^t + c_2e^{2t} + c_3e^{3t} = y_c$

Look for $y = c_1t + c_2$

Then $y' = c_1$ & $y'' = 0$, so we only have to solve

$11y' - 6y = 3t$

$11(c_1) - 6(c_1t + c_2) = 3t$

$11c_1 - 6c_1t - 6c_2 = 3t \longrightarrow$

$11c_1 - 6c_2 = 0$ &

$-6c_1 = 3 \longrightarrow$

$\boxed{c_1 = \frac{1}{2}}$ &

$11(\frac{1}{2}) - 6c_2 = 0$

$\frac{11}{2} = -6c_2$

$\boxed{-\frac{11}{12} = c_2}$

So $y_p = -\frac{1}{2}t - \frac{11}{2}$

& $y = y_c + y_p$ is the general solution.

Particular Solutions will be sort of tricky and sort of easy at the same time.

MOAR Superposition Principle!!!

Theorem 4.1.7 Superposition Principle—Nonhomogeneous Equations

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be k particular solutions of the nonhomogeneous linear nth-order differential equation (7) on an interval I corresponding, in turn, to k distinct functions $g_1, g_2, \ldots, g_k$. That is, suppose y_{p_i} denotes a particular solution of the corresponding differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_i(x), \tag{12}$$

where $i = 1, 2, \ldots, k$. Then

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x) \tag{13}$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y$$
$$= g_1(x) + g_2(x) + \cdots + g_k(x). \tag{14}$$

$L(y) = \sin(x) - \cos(x)$ Find y_p for $\sin(x)$ & add them

Find y_p for $\cos(x)$

This is basically saying that if you have a big, messy right hand side that has a whole bunch of functions added together, you can find a particular solution for each summand and find a particular solution to the original nonhomogeneous equation by adding them together!

$L(y) = g_1(x) + g_2(x)$

Find y_{p_1} for $g_1(x)$ & y_{p_2} for $g_2(x)$ &

Let $y_p = y_{p_1} + y_{p_2}$.

The "L" stands for any linear operator, but it will be the differential operator given by the left-hand-side of the differential equation.

$$L(y_{p_1} + y_{p_2}) = \underbrace{L(y_{p_1})}_{=\sin(x)} + \underbrace{L(y_{p_2})}_{=-\cos(x)} = g_1(x) + g_2(x) \checkmark$$