Nonlinear Models

Uninhibited Growth - Exponential $P = P_0 e^{kt}$

Limits to Growth - Logistic Model. A bit more realistic, but still never once found in Nature. Usually the jumping-off point for real models.

$$\frac{\frac{dP}{dt}}{P} = K$$

Let *K* = the carrying capacity of the environment.

$P = P(t)$ = Population as a function of time, t.

Instead of $P'(t) = KP(t)$, (2)

we look at $P'(t) = f(P)P(t)$, where $f(0) = r$, $f(K) = 0$,

so growth rate $\to 0$ as $P \to K$arrying capacity.

Assume $f(P)$ is linear. Then

$(0, r), (K, 0) \longrightarrow m = \frac{-r}{K}$ &

$y = -\frac{r}{K}(P - 0) + r = -\frac{r}{K}P + r \Longrightarrow$

$$\frac{dP}{dt} = \left(-\frac{r}{K}P + r\right)P \qquad (3)$$

$= \frac{r}{K}(K - P)P$ is how I see it.

But Book relabels

$$\frac{dP}{dt} = (a - bP)P \text{ or } P(a - bP) \qquad (4)$$

Verhulst: $a, b > 0$. (ca 1840)

(4) is LOGISTIC EQN

$$\frac{dP}{dt} = aP - bP^2$$

P is bounded above as $t \to \infty$.

$$\frac{dP}{dt} = aP - bP^2 = P(a-bP)$$

Separation of Variables

$$\frac{dP}{aP-bP^2} = dt$$

$$\int \frac{dP}{aP-bP^2} = \int dt = t + C$$

$$\frac{1}{P(a-bP)} = \frac{A}{P} + \frac{B}{a-bP}$$

$$\Rightarrow \quad 1 = A(a-bP) + BP$$

$P=0 \Rightarrow$

$1 = aA \Rightarrow \boxed{A = \frac{1}{a}}$

$P = \frac{a}{b} \Rightarrow \quad 1 = B\left(\frac{a}{b}\right) \longrightarrow \boxed{B = \frac{b}{a}} \longrightarrow$

$$\left(\frac{1/a}{P} + \frac{b/a}{a-bP}\right)dP = dt$$

$$= \left(\frac{\frac{1}{a}}{P} - \frac{\frac{b}{a}}{bP-a}\right)dP$$

$$= \left(\frac{\frac{1}{a}}{P} - \frac{\frac{1}{a}}{P-\frac{a}{b}}\right)dP$$

$$\frac{1}{P(a-bP)} = \frac{\frac{1}{a}}{P} + \frac{\frac{b}{a}}{a-bP}$$

$$= \frac{1}{a}\left(\frac{1}{P} + \frac{b}{a-bP}\right)$$

$$= \frac{1}{a}\left(\frac{1}{P} - \frac{b}{bP-a}\right) = \frac{1}{a}\left(\frac{1}{P} - \frac{1}{P-\frac{a}{b}}\right)$$

$$\frac{1}{a}\int \frac{dP}{P} - \frac{1}{a}\int \frac{b}{bP-a}\,dP$$

$$\frac{1}{a}\ln|P| - \frac{1}{a}\ln|a-bP| = t + c$$

Not getting this quite, but

$|a-bP| = |bP-a|$

$= |-1(a-bP)|$

$= |-1|\,|a-bP| = |a-bP|$

$$= \frac{1}{a}\left[\ln|P| - \ln|bP-a|\right]$$

$$= \frac{1}{a}\left[\ln\left|\frac{P}{bP-a}\right|\right] = \frac{1}{a}\ln\left|\frac{P}{a-bP}\right|$$

Integrating RHS & equating:

$$\frac{1}{a}\ln\left|\frac{P}{a-bP}\right| = \int dt = t + C$$

$$\ln\left|\frac{P}{a-bP}\right| = at + aC \Longrightarrow e^{at+aC} = e^{at}e^{aC} = e^{at}c_1$$

$$\frac{P}{a-bP} = c_1 e^{at}, \text{ where } c_1 = e^{aC}.$$

$$P = c_1 e^{at}(a-bP) = ac_1e^{at} - bc_1Pe^{at} \Longrightarrow$$

$$P + bc_1Pe^{at} = P(1+bc_1e^{at}) = ac_1e^{at}$$

$$P = \frac{ac_1e^{at}}{1+bc_1e^{at}}$$

$$P(0) = P_0 = \frac{ac_1}{1+bc_1} \quad \text{Solve for } c_1 \text{ (why?)}$$

$$P_0(1+bc_1) = P_0 + bP_0c_1 = ac_1$$

$$bP_0c_1 - ac_1 = -P_0$$

$$c_1(bP_0 - a) = -P_0$$

$$c_1 = \frac{-P_0}{bP_0-a} = \frac{P_0}{a-bP_0}$$

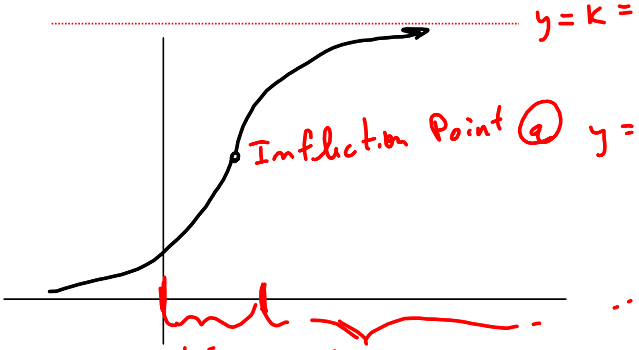
$$P = \frac{ac_1e^{at}}{1+bc_1e^{at}} = \frac{\frac{aP_0}{a-bP_0}e^{at}}{1+\frac{P_0}{a-bP_0}be^{at}} = \frac{\frac{aP_0}{\cancel{a-bP_0}}e^{at}}{\frac{a-bP_0+bP_0e^{at}}{\cancel{a-bP_0}}}$$

$$= \frac{aP_0e^{at}}{(a-bP_0)+bP_0e^{at}} = \boxed{\frac{aP_0}{bP_0+\underbrace{(a-bP_0)e^{-at}}_{\text{Transitory}}} = P(t)}$$

Top & Bottom $\div\, e^{at}$

$$t \to \infty \quad \frac{aP_0}{bP_0} = \frac{a}{b} = K$$

General Behavior



Click here for a direct solution, by differentiating the *P*(*t*) we found and setting *P''*(t) = 0

Book Method:

$$\frac{dP}{dt} = aP - bP^2 \Rightarrow$$

$$\frac{d^2P}{dt^2} = a\frac{dP}{dt} - 2bP\frac{dP}{dt} = (a-2bP)\big(\underbrace{P(a-bP)}_{\frac{dP}{dt}}\big) \overset{\text{SET}}{=} 0 \Rightarrow$$

$$= (a-2bP)\frac{dP}{dt}$$

$$P = \frac{a}{2b},\ P = 0,\ P = \frac{a}{b}$$

$P = \frac{a}{2b}$: Yep.

$P = 0,\ P = \frac{a}{b}$: Nope

Variations on Logistic:

Add/subtract some number or function from the RHS.

In my dissertation, I used something similar that I called my "Marlin Perkins" term.

$$\frac{dP}{dt} = P(a-bP) + h$$

Rate of Harvest is proportional to Pop:

$$= P(a-bP) - cP$$

Immigration (small compared to P)

$$P' = P(a-bP) + ce^{-kP}$$

Gompertz $P' = P(a - b\ln(P))$

~~Die Down~~

Stop Growing sooner than $P(a-bP)$

Chemical Reactions Suppose that a grams of chemical A are combined with b grams of chemical B. If there are M parts of A and N parts of B formed in the compound and $X(t)$ is the number of grams of chemical C formed, then the number of grams of chemical A and the number of grams of chemical B remaining at time t are, respectively,

$$\text{Amt } A = a - \frac{M}{M+N}X$$

$$\text{Amt } B = b - \frac{N}{M+N}X$$

The law of mass action states that when no temperature change is involved, the rate at which the two substances react is proportional to the product of the amounts of A and B that are untransformed (remaining) at time t:

$$\frac{dX}{dt} \qquad X = \hat{k}\left(a - \frac{M}{M+N}X\right)\left(b - \frac{N}{M+N}X\right) \qquad (8)$$

$$= \hat{k}\left(\frac{M}{M+N}\right)\left(\frac{N}{M+N}\right)(\alpha - X)(\beta - X), \text{ where} \qquad (9)$$

$$\alpha = a\left(\frac{M+N}{M}\right), \quad \beta = b\left(\frac{M+N}{N}\right)$$