§2.5

*Each DE in Problems 1–14 is homogeneous.*

In Problems 1–10 solve the given differential equation by using an appropriate substitution.

1. $(x - y)\,dx + x\,dy = 0$
2. $(x + y)\,dx + x\,dy = 0$
3. $x\,dx + (y - 2x)\,dy = 0$
4. $y\,dx = 2(x + y)\,dy$
5. $(y^2 + yx)\,dx - x^2\,dy = 0$
6. $(y^2 + yx)\,dx + x^2\,dy = 0$
7. $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$
8. $\dfrac{dy}{dx} = \dfrac{x + 3y}{3x + y}$
9. $-y\,dx + \left(x + \sqrt{xy}\right)dy = 0$
10. $x\dfrac{dy}{dx} = y + \sqrt{x^2 - y^2}, \quad x > 0$

① $(x-y)\,dx + x\,dy = 0$

$x\,dy = (y-x)\,dx$

$$\frac{dy}{dx} = \frac{y-x}{x} = \frac{y}{x} - 1$$

$$\frac{dy}{dx} - \frac{1}{x}y = -1$$

Linear $1^{st}$-order

I.F. $\mu = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$, etc.

(10) $x\frac{dy}{dx} = y + \sqrt{x^2 - y^2}$

$x\,dy + (-(y + \sqrt{x^2 - y^2}))dy = 0$

$M(x,y) = x$ $\qquad N(x,y) = -(y + \sqrt{x^2 - y^2})$

$M(tx,ty) = t^1 M \quad \alpha = 1 \qquad N(tx,ty) = -(ty + \sqrt{(tx)^2 - (ty)^2})$

$= t(-(y + \sqrt{x^2 - y^2}))$ (if $t \geq 0$)

Let $u = \frac{y}{x} \Longrightarrow$

$y = ux \Longrightarrow$

$M(x,y) = xM(1,u)$ & $N(x,y) = xN(1,u)$

$= x(1)$ $\qquad = x(-(1 + \sqrt{1-u^2}))$

$\Longrightarrow dx + (-(1 + \sqrt{1-u^2}))(u\,dx + x\,du) = 0$

$\Longrightarrow dx - (u + u\sqrt{1-u^2})dx - (x + x\sqrt{1-u^2})du = 0$

$\Longrightarrow (1 - u - u\sqrt{1-u^2})dx = x(1 + \sqrt{1-u^2})du$

$\frac{dx}{x} = \frac{1 + \sqrt{1-u^2}}{1 - u - u\sqrt{1-u^2}}du$

E1

§2.6

$y' = 0.1\sqrt{y} + .4x^2 \quad y(2)=4$

$x_0 = 2, y_0 = 4$

$f(x,y) = .1\sqrt{y} + .4x^2$

$(x_0, y_0) = (2,4)$

$x_1 = x_0 + h$

$y(x_1) \approx y_0 + y'(x_0)(x_1 - x_0)$

$= y_0 + f(x_0,y_0)(h) = hf(x_0,y_0) + y_0$

$y(x_2) \approx y(x_1) + f(x_0,y_0)h$

$y(x_{n+1}) \approx y(x_n) + y'(x_n)\underbrace{(x_{n+1} - x_n)}_{h}$

$= y(x_n) + hf(x_n,y_n)$

$= hf(x_n,y_n) + y(x_n)$

DOWNLOAD EXCEL IMPLEMENTATION BY CLICKING HERE

**Book Typo:**

**On page 79, it says "In Example 1, the true or actual values were calculated..."**

**They mean Example 2, because the "known solution" is not a solution for the equation in Example 1.**

**It says "Verify," and of course, it's the solution to Example 2.**

Tangent line

$L(x) = f'(x_0)(x - x_0) + f(x_0)$

In Problems 5–10 use a numerical solver and Euler's method to obtain a four-decimal approximation of the indicated value. First use $h = 0.1$ and then use $h = 0.05$.

9. $y' = xy^2 - \dfrac{y}{x}, \quad y(1) = 1; \quad y(1.5)$

**Use my spreadsheet template to work this one.**

**Section 3.1 - Linear Models**

Uninhibited Growth

$y'$ is proportional to $y$.

$y' = ky$ for some real $k$.

$$\frac{y'}{y} = \frac{\frac{dy}{dt}}{y} = k \implies$$

$$\frac{dy}{y} = k\,dt \implies$$

$$\ln|y| = \ln(y) = kt + C$$

$$y = e^{kt+C} = e^{kt}e^{C} = \hat{C}e^{kt}$$

Notice That $y(0) = \hat{C}e^{0} = \hat{C}$.

So if $t=0$ corresponds to start,

then $y(t) = y(0)e^{kt}$
$= y_0 e^{kt}$

Pop. is 30,000 in 1978. Its relative growth rate is 3%.

~~Then $y = 30{,}000e^{kt}$~~

Growth rate is 3%.

Then $y =$

$$y' = .03y$$

$$\frac{y'}{y} = .03$$

$$\ln(y) = .03t + C$$

$$y = \hat{C}e^{.03t}$$

where $\hat{C} = y(0)$, since ... (above)

**13.** Archaeologists used pieces of burned wood, or charcoal, found at the site to date prehistoric paintings and drawings on walls and ceilings of a cave in Lascaux, France. See Figure 3.1.12. Use the information on page 87 to determine the approximate age of a piece of burned wood, if it was found that 85.5% of the C-14 found in living trees of the same type had decayed.

Rock painting showing a horse and a cow, c.17000 BC (cave painting)/Prehistoric/Didier Lenart Archive/Caves of Lascaux, Dordogne, France/Bridgeman Images

**Figure 3.1.12** Cave wall painting in Problem 13

100 − 85.5 = 14.5% of radioactive C-14 remains.

$\frac{1}{2}$-life is 5730 yrs.

$P_0 e^{5730k} = \frac{1}{2} P_0$

$e^{5730k} = \frac{1}{2}$

$5730k = \ln\left(\frac{1}{2}\right) = \ln(2^{-1})$

$= -\ln(2) \longrightarrow$

$k = \frac{-\ln(2)}{5730}$

14.5% is left:

$P_0 e^{kt} = 14.5\% \, P_0$

$e^{kt} = .145$

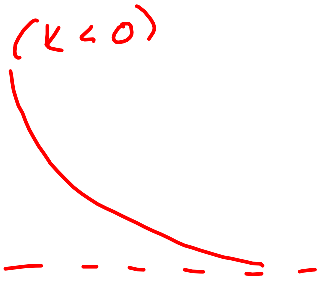
$\cancel{kt = \frac{.145}{}}$

$\cancel{t = \frac{.145}{k}}$

$\cancel{= \frac{.145}{\frac{-\ln(2)}{5730}} = \frac{.145(5730)}{-\ln(2)}}$

$P(5730) = \frac{1}{2} P_0$

$P(t) = P_0 e^{kt}$




$e^{kt} = .145$

$kt = \ln(.145)$

$t = \frac{\ln(.145)}{k} = \frac{\ln(.145)\,5730}{-\ln(2)} \approx$ 15963.06487

**Example 4 - Newton's Law of Cooling.**

**"The rate of cooling is proportional to the difference betweeen the temperature of the cake and the room temperature."**

Let $T = T(t) =$ Temperature (degrees F (or C))

& $T_0 =$ Room Temp.

Then $T'(t) = K(T(t) - T_0)$ for some $K$

$$\frac{dT}{T - T_0} = K\,dt \quad \cdots \hat{C}e^{K(T-T_0)}$$

§3.1 #20 what if $T_0 = T_A(t) =$ Ambient Temp, which changes over time.

Then $T_c'(t) = K(T_c(t) - T_A(t))$

$\Rightarrow T_c' - KT_c = -kT_A$ is linear 1st order ODE, with integrating factor $e^{-kt}$

$$\mu = e^{\int -k\,dt} = \text{integrating factor}$$

**Example 5 - Brine Mixtures**

$V \doteq$ Volume $= 300$ gals

Pumping salt with concentration $2\,\text{lb}/\text{gal}$

at a rate of $3 \frac{\text{gal}}{\text{min}}$

Pumping salt mixture out @ $\frac{3\text{gal}}{\text{min}}$

Let $A = A(t)$ = Amt of salt in the tank (in lbs)

**Assume instant, perfect mixing (LOL!).**

SALT RATE IN $= R_I = \left(3 \frac{\text{gal}}{\text{min}}\right)\left(\frac{2\text{lb}}{\text{gal}}\right) = 6 \frac{\text{lb}}{\text{min}}$

SALT RATE OUT $= \left(A(t)\text{ lb}\right)\left(\frac{1}{300\text{gal}}\right)\left(3 \frac{\text{gal}}{\text{min}}\right) = \frac{A(t)}{100} \frac{\text{lb}}{\text{min}} = R_O$

$\frac{dA}{dt} = R_I - R_O$ (if conc in is > conc. of tank)

$= 6 - \frac{A(t)}{100} \rightarrow$

$A' + \frac{1}{100}A = 6$

$e^{\frac{1}{100}t}A' + e^{\frac{1}{100}t}A\left(\frac{1}{100}\right) = 6e^{\frac{1}{100}t}$

$\int d\left(e^{\frac{1}{100}t}A\right) = \int 6e^{\frac{1}{100}t}\,dt$

$e^{\frac{t}{100}}A = 600e^{\frac{1}{100}t} + c$

$A = 600 + ce^{-\frac{1}{100}t}$

$A(0) = 50 \rightarrow 600 + c = 50 \rightarrow$

$c = -550 \rightarrow$

$$\boxed{A(t) = 600 - 550e^{-\frac{1}{100}t}}$$

**Series Circuits - Moar Linear ODEs.**

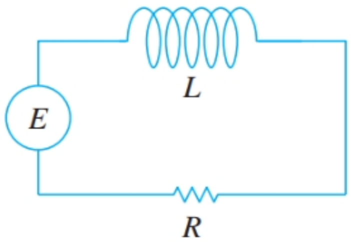



Fig 3.1.9

Kirchhoff's second law states that the sum of the voltage drop across the inductor ($L(di/dt)$) and the voltage drop across the resistor ($iR$) is the same as the impressed voltage ($E(t)$) on the circuit. See Figure 3.1.9.

$$L\frac{di}{dt} + Ri = E$$

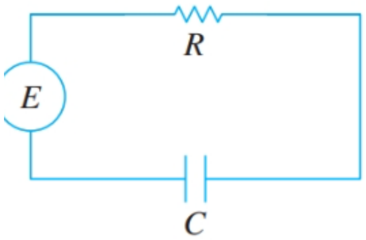



Fig 3.1.10

The voltage drop across a capacitor with capacitance $C$ is given by $q(t)/C$, where $q$ is the charge on the capacitor. Hence, for the series circuit shown in Figure 3.1.10, Kirchhoff's second law gives

$$Ri + \frac{1}{C}q = E(t)$$

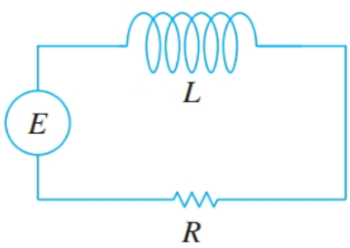
$$i = \frac{dq}{dt} \Rightarrow R\frac{dq}{dt} + \frac{1}{C}q = E$$

current

change in charge wrt time.

31. A 30-volt electromotive force is applied to an $LR$-series circuit in which the inductance is 0.1 henry and the resistance is 50 ohms. Find the current $i(t)$ if $i(0) = 0$. Determine the current as $t \to \infty$.

$$0.1\frac{di}{dt} + 50i = 30$$




$$L\frac{di}{dt} + Ri = E$$

Kirchhoff's second law states that the sum of the voltage drop across the inductor ($L(di/dt)$) and the voltage drop across the resistor ($iR$) is the same as the impressed voltage ($E(t)$) on the circuit. See Figure 3.1.9.