§2.4 Suppose μ makes $M\,dx + N\,dy = 0$ exact

i.e. $\mu M\,dx + \mu N\,dy = 0$ is exact.

Then $(\mu M)_y = (\mu N)_x$

$\mu_y M + \mu M_y = \mu_x N + \mu N_x$

$\mu_y M - \mu_x N = \mu N_x - \mu M_y$

$\boxed{\text{Assume } \mu = \mu(x)}$ → If this doesn't work, Assume $\mu = \mu(y)$. Then μ_x drops out.

Then $-\mu_x N = \mu (N_x - M_y)$

$$\frac{-\mu_x}{\mu} = \frac{N_x - M_y}{N}$$

$$\frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}$$

$$\ln|\mu| = \int \frac{M_y - N_x}{N}\,dx$$

$$|\mu| = e^{\int \frac{M_y - N_x}{N}\,dx}$$

$$\mu = \pm e^{\int \frac{M_y - N_x}{N}\,dx}$$ & either will work.

$$\mu = e^{\int \frac{M_y - N_x}{N}\,dx}$$

$$\frac{\mu_x}{\mu} = \frac{\frac{\partial \mu}{\partial x}}{\mu}$$

Section 2.5 - Solutions by Substitutions

S $\frac{dy}{dx} = f(x,y)$, where $f(x,y)$ is "ugly."

we make the substitution $y = g(x,u) = g(x, u(x))$.

Then $\frac{dy}{dx} = \frac{d\,g(x,u)}{dx} = \frac{\frac{\partial g}{\partial x}\cdot dx + \frac{\partial g}{\partial u} du}{dx} = \frac{\partial g}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial g}{\partial u}\cdot\frac{du}{dx}$

$= \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\cdot\frac{du}{dx} \Longrightarrow \boxed{dy = dg + \frac{\partial g}{\partial u} du}$

$\Longrightarrow \frac{\partial g}{\partial u}\cdot\frac{du}{dx} = \frac{dy}{dx} - \frac{\partial g}{\partial x}$

$u = \frac{y}{x}$

$\Longrightarrow \frac{du}{dx} = \frac{\frac{\partial y}{\partial x} - \frac{\partial g}{\partial x}}{\frac{\partial g}{\partial u}} = F(x,u)$

OR $v = \frac{x}{y}$

If we can solve for u, then we can solve for y.

There are 3 different kinds of 1st-order differential equations that are solvable by this method:

1. Homogeneous Equations.

S $f(tx, ty) = t^{\alpha} f(x,y)$. Then f is a homogeneous function of degree α.

$\boxed{E}$ $f(x,y) = x^3 + y^3$

$f(tx,ty) = (tx)^3 + (ty)^3 = t^3(x^3+y^3) = t^3 f(x,y)$

is hom. of degree $\alpha = 3$.

$f(x,y) = \frac{x-5y}{4x+2y}$ is hom. of degree zero. $f(tx,ty) = \frac{t}{t} f(x,y) = t^0 f(x,y)$

$g(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ is hom. of degree $\alpha = -1$

In differential form

$M dx + N dy = 0$ is a hom. eq'n of degree α

if $M(tx,ty) = t^{\alpha} M(x,y)$ & $N(tx,ty) = t^{\alpha}(x,y)$

with same α for each.

FACT: If M, N hom. of degree α, then the substitution $u = \frac{y}{x}$ gives $y = ux \Longrightarrow$

$M(x,y) = M(x, ux) = M(x \cdot 1, xu) = x^{\alpha} M(1, u)$

& $N(x,y) = N(x, ux) = x^{\alpha} N(1, u)$

$v = \frac{x}{y}$ is also legit.

Trick: If M is simpler: $v = \frac{x}{y}$
N is " : $u = \frac{y}{x}$

Let $v = \frac{x}{y} \Longrightarrow$

$M(x,y) = y^{\alpha} M(v, 1)$ & $N(x,y) = y^{\alpha} N(v, 1)$

This will make the eq'n separable.

Now, if $u = \frac{y}{x}$

Then $y = ux \Rightarrow$

$M(x,y)\,dx + N(x,y)\,dy$

$= x^d M(1,u)\,dx + x^d N(1,u)\,dy$

$u = \frac{y}{x}$

$y = xu \Rightarrow$

$dy = \frac{\partial(xu)}{\partial x}\,dx + \frac{\partial(xu)}{\partial u} \cdot du$

$= u\,dx + x\,du$

$d(xu) = u\,dx + x\,du$

$= x^d M(1,u)\,dx + x^d N(1,u)\,dy = 0$

$\Rightarrow M(1,u)\,dx + N(1,u)\,dy = 0$

$\Rightarrow M(1,u)\,dx + N(1,u)[x\,du + u\,dx]$

$\Rightarrow (M(1,u) + uN(1,u))\,dx + N(1,u)\,x\,du = 0$

Should be separable.

So, the method is

1. Make the substitution $u = \frac{y}{x} \Rightarrow y = ux$
2. Compute $dy = x\,du + u\,dx$ $(y = ux)$
3. This will give
   $(M(1,u) + uN(1,u))\,dx + xN(1,u)\,du$

For $u = \frac{x}{y}$, much the same, but things moved around

$\boxed{E1}$ $(x^2+y^2)dx + (x^2-xy)dy$

$M(tx,ty) = t^2x^2+t^2y^2 = t^2(x^2+y^2) = t^2 M(x,y)$ hom.

$N(tx,ty) = t^2x^2 - tx\,ty = t^2(x^2-xy) = t^2 N(x,y)$ $d=2$

$y = ux \Rightarrow dy = x\,du + u\,dx$

$\Rightarrow (x^2+u^2x^2)dx + (x^2-x(ux))dy$

$= x^2(1+u^2)dx + x^2(1-u)dy$

$= x^2\big((1+u^2)dx + (1-u)dy\big) = 0 \longrightarrow$

$(1+u^2)dx + (1-u)dy$

$= (1+u^2)dx + (1-u)(x\,du + u\,dx) = 0$

$\Rightarrow (1+u^2)dx + (1-u)x\,du + (1-u)u\,dx$

$= \left(1+u^2+(1-u)u\right)dx + x(1-u)\,du = 0$

$\Rightarrow (1+u)\,dx + x(1-u)\,du = 0$

$\Rightarrow (u+1)\,dx = x(u-1)\,du$

$\Rightarrow \frac{dx}{x} = \frac{u-1}{u+1}\,du$ separable!

$9\overline{)28}$ 3 r 1

This means $28 = 3(9)+1$

$\frac{28}{9} = 3+\frac{1}{9}$

$\int \frac{u-1}{u+1}\,du = \int\left(1 - \frac{2}{u+1}\right)du = u - 2\ln|u+1| + \hat{C}$

$\hat{C} = \ln|c|$

$f(x) = p(x)\,g(x) + r(x)$

$\frac{f(x)}{g(x)} = p(x) + \frac{r(x)}{g(x)}$

$-1 \,)\; 1 \quad -1$
$\quad\;\; -1$
$\overline{\quad 1 \quad -2 = r}$

$u-1 = (u+1)(1) - 2$

$\frac{u-1}{u+1} = 1 - \frac{2}{u+1}$

$\ln|x| = u - 2\ln|u+1| + \ln|c|$

$\ln|x| - u + 2\ln|u+1| = \ln|c|$

$\Rightarrow \left(e^{\ln|x|}\right)\left(e^{-u}\right)\left(e^{\ln(|u+1|^2)}\right) = |c|$

$|x|e^{-u}|u+1|^2 = c$ (if $c>0$)

$|x|e^{-\frac{y}{x}}(u+1)^2 = c$

$|x|e^{-\frac{y}{x}}\left(\frac{y}{x}+1\right)^2 = c$

$|x|e^{-\frac{y}{x}}\left(\frac{y+x}{x}\right)^2 = c$

$\frac{|x|}{x^2}e^{-\frac{y}{x}}(x+y)^2 = c$

$\frac{x^2}{|x|} = \begin{cases} \frac{x^2}{x} = x & \text{if } x>0 \\ \frac{x^2}{-x} = -x & \text{if } x<0 \end{cases}$

$\Rightarrow \begin{cases} (x+y)^2 = cxe^{\frac{y}{x}} & (x>0) \\ (x+y)^2 = -cxe^{\frac{y}{x}} & (x<0) \end{cases}$

Trick: If M is simpler: $v = \frac{x}{y}$

N is " : $u = \frac{y}{x}$

Bernoulli's Equation

$y' + P(x)y = f(x)y^n$

substitute $u = y^{1-n}$

This makes it a linear eq'n

Reduction to a Separable Equation

$\frac{dy}{dx} = f(Ax + By + C) \quad (B \neq 0)$

Let $u = Ax + By + C$