Solve the given initial-value problem by finding, as in Example 4, an appropriate integrating factor.

$$(x^2 + y^2 - 3)\,dx = (y + xy)\,dy, \quad y(0) = 1$$

$\to (x^2+y^2-3)dx + (-y-xy)dy = 0$

$\Rightarrow M = x^2+y^2-3, \; N = -y-xy$

$\mu M dx + \mu N dy = 0$ is exact

Book way

$M = x^2+y^2-3 \qquad N = -y-xy$

$M_y = 2y \qquad N_x = -y$

$(\mu M)_y = (\mu N)_x \Rightarrow$

$\mu_y M + \mu M_y = \mu_x N + \mu N_x$

$\mu_y(x^2+y^2-3) + \mu(2y) = \mu_x(-y-xy) + \mu(-y)$

$\mu_y(x^2+y^2-3) + \mu_x(y+xy) = -2\mu y - \mu y = -3\mu y$

$\dfrac{M_y - N_x}{N} = \dfrac{2y+y}{-y-xy} = \dfrac{3y}{y(-1-x)} = \dfrac{-3}{x+1}$

Assume $\mu = \mu(x)$:

$\mu_x(y(1+x)) = -3\mu y$

$\int \dfrac{\mu_x}{\mu} = \int \dfrac{-3}{x+1}$

$\ln|\mu| = -3\ln|x+1|$

$|\mu| = (x+1)^{-3}$

$\mu = \left(\dfrac{1}{x+1}\right)^3$

$\dfrac{d\mu/dx}{\mu} = \dfrac{-3}{x+1}$

$\int \dfrac{d\mu}{\mu} = \int\left(\dfrac{-3}{x+1}\right)dx$

$\dfrac{x^2+y^2-3}{(x+1)^3} = \hat{M}, \quad \hat{N} = \dfrac{-y-xy}{(x+1)^3} = \dfrac{-y(x+1)}{(x+1)^3} = \dfrac{-y}{(x+1)^2}$

$\hat{M}_y = \dfrac{2y}{(x+1)^3}, \; \hat{N}_x = \dfrac{-2(-y)}{(x+1)^3} = \hat{M}_y$ ✓

Hmm. yeah that worked.

$\dfrac{x^2+y^2-3}{(x+1)^3}dx + \dfrac{-y}{(x+1)^2}dy = 0$

Then Let $f(x,y) = \int M(x,y)dx + g(y)$

Then $f_y = \dfrac{d}{dy}\int M(x,y)dx + g'(y)$

$\Rightarrow \hat{N}(x,y) - \dfrac{d}{dy}\int \hat{M}(x,y)\,dx = g'(y)$

$\dfrac{-y}{(x+1)^2}\boxed{-\dfrac{\partial}{\partial y}\int \dfrac{x^2+y^2-3}{(x+1)^3}dx} = \dfrac{-y}{(x+1)^2} - \int \dfrac{\partial}{\partial y}\hat{M}(x,y)dx = g'(y)$

$\dfrac{x^2}{(x+1)^3} + \dfrac{y^2-3}{(x+1)^3}$

y^2-3 is constant wrt x (partial integration)

$\Rightarrow \dfrac{-y}{(x+1)^2} - \hat{N}(x,y) - h(x) = g'(y) \to h(x) = C = g'(y)$

& so $g(y) = Cy + K$

$\to K = 0$

$f(x,y) = \int \hat{M}(x,y)dx + Cy$

$= \int \dfrac{x^2+y^2-3}{(x+1)^3}dx = \int \dfrac{x^2}{(x+1)^3}dx + \int \dfrac{y^2-3}{(x+1)^3}dx$

$\left(u = x^2 \quad dv = (x+1)^{-3}dx \quad = uv - \int v\,du = -\tfrac{1}{2}\left(\dfrac{x^2}{(x+1)^2}\right) - \int -\tfrac{1}{2}(x+1)^{-2}(2x\,dx) \right)$

$du = 2x\,dx \quad v = -\tfrac{1}{2}(x+1)^{-2}$

$= -\tfrac{1}{2}\left(\dfrac{x^2}{(x+1)^2}\right)$ ugh! See Maple for this! $\int$ part1 $+ \int$ part2

$$\frac{\partial}{\partial y}\int \hat{M}(x,y)\,dx$$

$$= \int \frac{\partial}{\partial y} M(x,y)\,dx$$

$$= \int \frac{\partial}{\partial x} N(x,y)\,dx$$

$$= \int \partial N(x,y) = N(x,y) + \underset{C}{\overset{g(x)}{\updownarrow}}$$

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$$\mu = \frac{1}{(x+1)^3}$$

$$\frac{x^2+y^2-3}{(x+1)^3}\,dx + \frac{1}{(x+1)^3}(-xy-y)\,dy = 0$$