Section 2.4 - Exact Differential Equations

Recall? Given $f(x,y)$, $\frac{\partial f}{\partial x}$ is the partial derivative of f with respect to x, obtained by differentiating w.r.t. x while holding y constant.

$\frac{\partial f}{\partial x}$ is partial derivative of f $\overbrace{\text{with respect}}^{\text{wrt}}$ to x

Recall: The (total) differential of a function $z = f(x,y)$ whose partial derivatives exist is

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f_x dx + f_y dy. \text{ where } f_x \equiv \frac{\partial f}{\partial x},\ f_y \equiv \frac{\partial f}{\partial y}$$

$\boxed{\text{D2.4.1}}$ A 1^{st} order ODE $M(x,y)dx + N(x,y)dy = 0$ is <u>Exact</u> if $M(x,y)dx + N(x,y)dy = dz$ for some $z = f(x,y)$.

Definition 2.4.1 Exact Equation

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$ defined in R. A first-order differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation** if the expression on the left-hand side is an exact differential.

It's easy to build an exact differential equation just by taking the differential of any *f*.

$z = f(x,y) = x^2 + 2x\sin y + y^2$

$f_x = 2x + 2\sin y$

$f_y = 2x\cos y + 2y$

$f(x,y) = c$ will be an implicit solution to $M\,dx + N\,dy$ if $df = M\,dx + N\,dy$

Our Job: Build the $f(x,y)$

$\rightarrow dz = (2x + 2\sin y)\,dx + (2x\cos y + 2y)\,dy = 0$

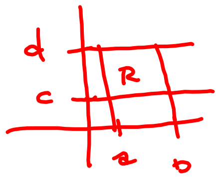
is an exact differential equation

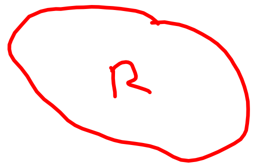
w/ $M(x,y) = 2x + 2\sin y$ & $N(x,y) = 2x\cos y + 2y$

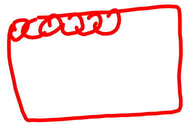
Theorem 2.4.1 Criterion for an Exact Differential

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{4}$$







From Calculus III (Taken from Stewart's Calculus, Section 14.3):

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Mixed 2nd partials of a twice-continuously differentiable function are the same.

Disk, or rectangle.... Much the same idea of a convex, bounded, connected region.

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$f(x,y) = x^2 + 2y\sin(x)$$

$$f_x = 2x + 2y\cos(x) \longrightarrow f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2\cos(x)$$

$$f_y = 2\sin(x) \longrightarrow f_{yx} = 2\cos(x)$$

Theorem 2.4.1 Criterion for an Exact Differential

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is **"Euler's Criterion for Exactness."**

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \qquad (4)$$

$M(x,y)\,dx + N(x,y)\,dy$ is an exact differential if and only if $M_y = N_x$

($M\,dx + N\,dy = 0$ exact iff $M_y = N_x$)

"if and only if" means the 1st implies the 2nd and the 2nd implies the 1st. Sometimes written "$\Longleftrightarrow$" or "iff"

"Necessity" "$\Longrightarrow$" "Necessary and sufficient"

Assume $M(x,y)\,dx + N(x,y)\,dy$ is an exact differential for some $f(x,y)$ that has cnts 1st & 2nd partials.

Then $f_x = M(x,y)$ & $f_y = N(x,y)$ and

$$f_{xy} = \frac{\partial M}{\partial y} = f_{yx} = \frac{\partial N}{\partial x}.$$

Conversely "sufficiency" "$\Longleftarrow$"

Suppose $M_y = N_x$.

We construct $f(x,y) \ni df = M(x,y)\,dx + N(x,y)\,dy$

This will suggest a method of solution.

$M(x,y) = 2x$

Then $f(x,y) = x^2 + y^2 - \sin(y) + \cos^2(2y)$

Let $f(x,y) \ni f_x = M(x,y)$

Then $f(x,y) = \int M(x,y)\,dx + g(y)$ $\forall$ diffble $g(y)$, ~~in particular $g(y) \equiv$ constant.~~

and so $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y)$ $\quad \frac{dg}{dy} = \frac{\partial g}{\partial y}$

$\Rightarrow f_y - \frac{\partial}{\partial y}\int M(x,y)\,dx = g'(y)$

i.e. $\boxed{N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx = g'(y)}$

Now, M & N are known, so these objects are all explicit and evaluable (usually).

& so $g(y) = \int N(x,y)\,dy - \int M(x,y)\,dy$ & satisfies

$z = f(x,y) = \int M(x,y)\,dx + g(y)$ $\quad =$

$dz = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

Since $df = M(x,y)\,dx + N(x,y)\,dy = 0$, we have $\boxed{f(x,y) = c}$ is an implicit soln.

$$= \frac{\partial}{\partial x}\left[\int M(x,y)\,dx + g(y)\right]dx + \frac{\partial}{\partial y}\left[\int M(x,y)\,dx + g(y)\right]dy$$

$$= \left(\frac{\partial}{\partial x}\int M(x,y)\,dx - \frac{\partial}{\partial x}g(y)\right)dx + \left(\frac{\partial}{\partial y}\int M(x,y)\,dx + \frac{\partial}{\partial y}g(y)\right)dy$$

$$= \left(\underset{\text{FTC I}}{M(x,y)} + 0\right)dx + \left[\frac{\partial}{\partial x}\int M(x,y)\,dx + g'(y)\right]dy$$

$$= M(x,y)\,dx + \left[\frac{\partial}{\partial y}\int M(x,y)\,dx + N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx\right]dy$$

$= M(x,y)\,dx + N(x,y)\,dy$ = Differential of the $f(x,y)$ we built.

$A \Rightarrow B$

If it rains, I take an umbrella.

The converse

$B \Rightarrow A$ If I take an umbrella, it rains.

Just because ~~I~~ take an umbrella, doesn't mean it's raining

$M\,dx + N\,dy = 0$ is exact

iff

$M_y = N_x$

" $\Rightarrow$ " Necessity:

$M\,dx + N\,dy = 0$ is exact D.E.

$\Rightarrow$ $M = f_x$ & $N = f_y$ for some $f(x,y)$

$\Rightarrow$ $M_y = f_{xy} = f_{yx} = N_x$

" $\Leftarrow$ " Sufficiency

$\boxed{E1}$ $2xy\,dx + (x^2-y)\,dy = 0$

$M(x,y) = 2xy$, $N(x,y) = (x^2-y)$

$\Rightarrow$ $M_y = 2x$, $N_x = 2x \Rightarrow$ EXACT

$f(x,y) = \int M(x,y)\,dx + g(y) = \int 2xy\,dx + g(y)$

$= x^2y + g(y)$

& $f_y(x,y) = x^2 + g'(y) = x^2 - 1 = N(x,y)$

$\Rightarrow g'(y) = -1$

$\Rightarrow g(y) = -y + \tilde{c} = -y$ if we let $\tilde{c} = 0$ &

why wouldn't we?

So $f(x,y) = x^2y - y = C$ is implicit soln

$y(x^2-1) = C$

$\Rightarrow$ $y = \frac{C}{x^2-1}$ is explicit soln.

$\frac{dy}{dx} = \frac{d}{dx}\left[C(x^2-1)^{-1}\right] = -C(x^2-1)^{-2}(2x) = \frac{-2Cx}{(x^2-1)^2}$

$dy = \frac{-2Cx}{(x^2-1)^2}\,dx$

$\Rightarrow$ $2xy\,dx + (x^2-1)\,dy$

$2x\left(\frac{C}{x^2-1}\right)dx + (x^2-1)\left(\frac{-2Cx}{(x^2-1)^2}\right)dx$

$= \frac{2Cx}{x^2-1}\,dx + \frac{-2Cx}{x^2-1}\,dx = 0$ ✓

Example 4: A non-exact equation made exact by clever choice of integrating factor.

$xy\,dx + (2x^2+3y^2-20)\,dy = 0$

Suppose $\mu(x)$ times the above makes it exact.

$\mu xy\,dx + \mu(2x^2+3y^2-20)\,dy = 0$ $\mu = \mu(x)$ or $\mu(y)$, depending.

Then $(\mu xy)_y = (\mu(2x^2+3y^2-20))_x$

This means that

$\mu_y(xy) + \mu x = \mu_x(2x^2+3y^2-20) + \mu(4x)$ $(fg)' = g'f + f'g$

$\Rightarrow \mu_y(xy) - \mu_x(2x^2+3y^2-20) = 4\mu x - \mu x = 3\mu x$ $(fg)' = f'g + fg'$

Assume $\mu = \mu(x)$. Then $\mu_y = 0$ & $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$-\mu_x(2x^2+3y^2-2) = 3\mu x$

$-\frac{d\mu}{dx} = \frac{3\mu x}{2x^2+3y^2-2}$ "still @ impasse."

S $\mu = \mu(y)$. Then $\mu_x = 0$ &

$\mu_y(xy) + \mu x = \mu_x(2x^2+3y^2-20) + \mu(4x)$ becomes

$xy\mu_y + \mu x = 4x\mu \Rightarrow$

$y\mu_y + \mu = 4\mu \Rightarrow$

$y\mu_y = 3\mu \Rightarrow$

$y\,d\mu = 3\mu\,dy$

~~$\Rightarrow y = 3y\mu$~~

$\Rightarrow \frac{\partial\mu}{\mu} = \frac{3dy}{y}$

$\Rightarrow e^{\ln\mu} = e^{3\ln y}$

$\mu = e^{3\ln|y|} = (e^{\ln|y|})^3 = y^3$

So,

$y^3\left[xy\,dx + (2x^2+3y^2-20)\,dy = 0\right]$

$\Rightarrow xy^4\,dx + (2x^2y^3+3y^5-20y^3)\,dy = 0$

$M_y = 4xy^3 \quad N_x = 4xy^3$ ✓

Let $f(x,y) \ni f_x(x,y) = M = xy^4$.

Then $f(x,y) = \int xy^4\,dx + g(y) = \frac{1}{2}x^2y^4 + g(y)$

& $f_y(x,y) = \frac{\partial}{\partial y}\int xy^4\,dx + g'(y)$

$= \frac{\partial}{\partial y}\left(\frac{1}{2}x^2y^4\right) + g'(y)$

$= 2x^2y^3 + g'(y) = N(x,y) =$

$= 2x^2y^3 + 3y^5 - 20y^3$

$\Rightarrow g'(y) = 3y^5 - 20y^3$

$\Rightarrow g(y) = \frac{1}{2}y^6 - 5y^4 + \hat{C}$ choose $\hat{C} = 0$.

$\Rightarrow f(x,y) = \frac{1}{2}x^2y^4 + g(y) = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = C$ is an implicit soln ■

Section 2.5 Substitution.

$$y(\ln(x) - \ln(y))\,dx = (x\ln(x) - x\ln(y) - y)\,dy$$

Not planning on going into the theory very deeply. Just going to follow our noses, intuitively, based on the *presumption* that an integrating factor exists, and proceeding to what it must be.

The rest is solving the resulting equation after the multiplication by the integrating factor.

The above is the 1st question on Chapter 2 Written Work.

I figured I could work at least the first question without 2.5, but no. Right out of the gate, they're giving you a 2.5 question!

$$y\left(\ln\left(\tfrac{x}{y}\right)\right)dx = \left(x\left(\ln\left(\tfrac{x}{y}\right)\right) - y\right)dy$$

$$-y\ln\left(\tfrac{y}{x}\right)dx = \left(-x\ln\left(\tfrac{y}{x}\right) + y\right)dy$$

$$\ln\left(\tfrac{x}{y}\right) = \ln\left(\left(\tfrac{y}{x}\right)^{-1}\right) = -1\ln\left(\tfrac{y}{x}\right)$$

Let $v = \frac{y}{x}$

Then $y = vx$

$$\frac{dy}{dx} = v'x + v = x\frac{dv}{dx} + v$$

$$-xv(\ln(v))\,dx = (x\ln(v) + xv)(v'x + v)\,dx$$

$$\cancel{-xv\ln(v)\,dx} = \left(-x^2v'\ln(v) \cancel{- xv\ln(v)} + x^2vv' + xv^2\right)dx$$

$$0 = \left(-x^2\frac{dv}{dx}\ln(v) + x^2v\frac{dv}{dx} + xv^2\right)dx$$

$$= -x^2\ln(v)\,dv + x^2v\,dv + xv^2\,dx = 0$$

$$\Rightarrow xv^2\,dx = x^2\ln(v)\,dv - x^2v\,dv$$

$$\Rightarrow xv^2\,dx = x^2(\ln(v) - v)\,dv$$

$$\frac{x}{x^2}\,dx = \frac{\ln(v) - v}{v^2}\,dv$$

$$\frac{dy}{dx} = v'x + v$$

$$dy = x\frac{dv}{dx}\cdot dx + v\,dx$$

Dropped a sign?

Here's my work from earlier. I think it's OK.

$-xv^2 dx = x^2(\ln(v)+v)\,dv \longrightarrow$ **Separable!**

$-\frac{1}{x}\,dx = \frac{\ln(v)+v}{v^2}\,dv$

$-\ln|x| = \int \ln(v)\left(\frac{1}{v^2}dv\right) + \int \frac{1}{v}\,dv = \int \frac{\ln(v)}{v^2}\,dv + \ln(v) + C$

$= \ln(v) + \int \frac{\ln(v)}{v^2}\,dv = \ln(v) + \ln(v)\left(-\frac{1}{v}\right) - \int -\frac{1}{v}\cdot\frac{1}{v}\,dv$

$\int u\,dg = ug - \int g\,du$ $\quad = \ln(v) - \frac{1}{v}\ln(v) + \int \frac{1}{v^2}\,dv$

$u = \ln(v)$ $\quad dg = \frac{1}{v^2}$ $\quad = \ln(v) - \frac{1}{v}\ln(v) - \frac{1}{v} + C$

$du = \frac{1}{v}\,dv$ $\quad g = -\frac{1}{v}$

$-\ln(x) = \ln\left(\frac{y}{x}\right) - \frac{1}{\frac{y}{x}}\ln\left(\frac{y}{x}\right) - \frac{x}{y} + C$

$-\ln(x) + C_1 = \ln\left(\frac{y}{x}\right) - \frac{x}{y}\ln\left(\frac{y}{x}\right) - \frac{x}{y}$