§2.2 #7

$(1+x^4)\,dy + x(1+4y^2)\,dx = 0$, s.t. $y(1)=0$

$\Rightarrow (1+x^4)\,dy = -x(1+4y^2)\,dx$

$\Rightarrow \int \frac{dy}{1+4y^2} = \int \frac{-x}{1+x^4}\,dx$

$\Rightarrow \int \frac{dy}{1+(2y)^2} = \int \frac{-x\,dx}{1+(x^2)^2}$

$\Rightarrow \frac{1}{2}\int \frac{2\,dy}{1+(2y)^2} = \frac{1}{2}\int \frac{-2x\,dx}{1+(x^2)^2}$

$\Rightarrow \int \frac{du}{1+u^2} = \int \frac{-du}{1+u^2}$

$(u=2y)$ $(u=x^2)$

$\Rightarrow \arctan(u) = -\arctan(u) + C$

Bad notation. Should have chosen a letter other than *u* on the right hand side. As long as we keep it straight, we should be good.

$\Rightarrow \arctan(2y) = -\arctan(x^2) + C$

$\tan(LHS) = \tan(RHS)$

$2y = \tan(-\arctan(x^2) + C) = \frac{\tan(-\arctan(x^2)) + \tan(C)}{1-\tan(-\arctan(x^2))\tan(C)}$

https://harryzaims.com/public_html/122/videos/chapter-02/test-2/cheat-sheet-test-2.pdf

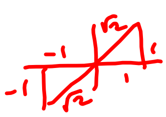
oops! Formula missing for $\tan(u+v) = \frac{\tan(u)+\tan(v)}{1-\tan(u)\tan(v)}$

$\Rightarrow 2y = \frac{-x^2+\tan(c)}{1+\tan(c)x^2} \Rightarrow y = \frac{1}{2}\left(\frac{-x^2+\tan(c)}{1+\tan(c)x^2}\right)$

Now, $y(1)=0 \Rightarrow$

$0 = \frac{1}{2}\left(\frac{-1^2+\tan(c)}{1+\tan(c)}\right) = \frac{1}{2}\left(\frac{\tan(c)-1}{\tan(c)+1}\right)$

$\Rightarrow 1-\tan(c) = 0 \Rightarrow \tan(c) = 1 \Rightarrow$



$\Rightarrow c = \frac{\pi}{4} + n\pi,\ n \in \mathbb{Z}$.

Finish: $y = \frac{1}{2}\left(\frac{\tan(c)-x^2}{1+\tan(c)x^2}\right) = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)$

Section 2.2 #10

Use a technique of integration or a substitution to find an explicit solution of the given differential equation.

$$\frac{dy}{dx} = \frac{\sin(\sqrt{x})}{\sqrt{y}}$$

$y =$ ✗

$$\Rightarrow \frac{2}{3}y^{\frac{3}{2}} = \int \sqrt{y}\,dy = \sin\sqrt{x}\,dx = 2\sin(\sqrt{x}) - 2\sqrt{x}\cos(\sqrt{x}) + c$$

$$\Rightarrow y^{\frac{3}{2}} = \frac{3}{2}\left[2\sin(\sqrt{x}) - 2\sqrt{x}\cos(\sqrt{x}) + c\right)$$

$$\Rightarrow y = \left(\frac{3}{2}\left[2\sin(\sqrt{x}) - 2\sqrt{x}\cos(\sqrt{x}) + c\right]\right)^{\frac{2}{3}}$$

Section 2.3 - Linear ODEs

Recall normal form of an ODE:

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}), \qquad (5)$$

where *f* is a continuous function of $x, y, y', y'', \ldots, y^{(n-1)}$

In 1.1, we said the special case of a *Linear n*[th]*-order ODE* can be written in the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \qquad (6)$$

where all the coefficients $a_0(x), a_1(x), \ldots, a_n(x)$ are cont^s functions of x, as is $g(x)$.

Special case of linear 1st-order and linear 2nd-order ODEs:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad \text{and} \quad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \qquad (7)^*$$

* from §1.1.

Definition 2.3.1 Linear Equation

A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad (1)$$

is said to be a **linear equation** in the variable y.

Standard Form: Divide through by the leading coefficient:

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)} \Longrightarrow$$

$$\frac{dy}{dx} + P(x)y = f(x), \text{ where } P(x) = \frac{a_0(x)}{a_1(x)} \text{ \& } f(x) = \frac{g(x)}{a_1(x)}$$

This is EQ(1), §2.3.

Method of solution:
write the left-hand side as the derivative of something, by adroit use of an "integrating factor" $\mu(x)$:

Want $\underbrace{\mu(x)\frac{dy}{dx} + \mu(x)P(x)\,y}_{\frac{d}{dx}[\mu(x)y] = \mu(x)y' + \mu'(x)y} = \mu(x)f(x)$

This means $\mu'(x)$ must be $\mu(x)P(x)$

$$\frac{d\mu}{dx} = \mu(x)P(x) \Longrightarrow$$

$$\frac{d\mu}{\mu(x)} = P(x)\,dx \Longrightarrow$$

$$\int\frac{d\mu}{\mu} = \int P(x)\,dx \Longrightarrow$$

$$\ln|\mu| = \int P(x)\,dx + C$$

$$\Longrightarrow |\mu| = e^{\int P(x)dx + C} = e^C e^{\int P(x)dx}$$

$$\Longrightarrow \mu = \pm e^C e^{\int P(x)dx} = ke^{\int P(x)dx}, \text{ where } k = \pm e^C$$

This works for any k, so why not $k = 1$? $\boxed{\mu = e^{\int P(x)dx}}$

$\mu(x)\frac{dy}{dx} + \mu(x)P(x)$ is thus

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)$$

$$= \frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}g(x)$$

$$\int d\left[e^{\int P(x)dx}y\right] = \int e^{\int P(x)dx}g(x)\,dx$$

$$\Rightarrow e^{\int P(x)dx}y = \int e^{\int P(x)}f(x)\,dx + C$$

$$\Rightarrow y = \frac{\int e^{\int P(x)dx}f(x)\,dx + C}{e^{\int P(x)dx}} = ce^{-\int P(x)dx} + e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx$$

Eqn (4)

This is the method of solution we will use.

It works for any linear 1st-order ODE, with all the cautions mentioned previously about the effect on domains, which can very much change when we go around dividing by functions to generate the standard forms and apply the above procedure.

While this always works, remember the techniques for Separable Equations, because they're easier, so always see if you can separate the variables before embarking on the integrating-factor adventure.

Book Method:

Solving a Linear First-Order Equation

(*i*) Remember to put a linear first-order equation into the standard form (2).

(*ii*) From the standard form of the equation identify $P(x)$ and then find the integrating factor $e^{\int P(x)\,dx}$. No constant need be used in evaluating the indefinite integral $\int P(x)dx$.

(*iii*) Multiply both sides of the standard form equation by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of the product of the integrating factor $e^{\int P(x)dx}$ and y:

$$\frac{d}{dx}\left[e^{\int P(x)\,dx}y\right] = e^{\int P(x)\,dx}f(x). \tag{5}$$

(*iv*) Integrate both sides of the last equation and solve for y.

What a wonderful, clunky way of laying it down. It is correct, but here's what you actually will do:

Given a Linear ODE:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

0. Check for separability with a little algebra skill. If it's separable, solve like a separable equation and ignore the instructions, below:

1. Divide the equation by the leading coefficient $a_1(x)$. This gives Standard Form:

$$\frac{dy}{dx} + P(x)y = f(x)$$

2. Evaluate the expressions $\int P(x)dx$, $e^{\int P(x)dx}$, & $\int e^{\int P(x)dx} f(x)\,dx$

3. Write the solution:

$$y = \frac{\int e^{\int P(x)dx} f(x)\,dx}{e^{\int P(x)dx}}$$

← This will have a constant of integration with it

The Book Method is written that way to reinforce the theoretical derivation of the method, which is good for your theoretical know-how, but not very efficient.

$$e^{-\int P(x)dx}\int e^{\int P(x)dx} f(x)\,dx + ce^{-\int P(x)dx}$$

↑ c is constant from above, that I mentioned, but didn't write.

Transient Term:

$T(t) = h(t) + ce^{-t}$

as $t \to \infty$, $ce^{-t} \to 0$.

ce^{-t} is a "transient" or "vanishing term.

Eventually, it goes away.

General Solution: $y' + P(x)y = f(x)$

Subject to the domains of $P(x)$ & $f(x)$

of continuity (Domain & interval(s) of continuity are the same, for most practical purposes.)

Fact: When the coefficient functions of a linear ODE are constants, the equation is autonomous, and you want to be on the lookout for critical points, which correspond to (constant) equilibrium solutions.

Page 58 - Mention is made of how it can be proved that

... if there's a solution, it looks like the one I wrote.

... conversely, if you have a function that looks like the one I wrote, then it is a solution.

The collection of functions of the form

$$y = ce^{-\int P(x)dx} + e^{-\int P(x)dx} \int e^{-\int P(x)dx} f(x)dx,$$

is the one-parameter family of solutions

'c' is the parameter.

$$\text{Sol'n set} = \left\{ e^{-\int P(x)dx} \int e^{-\int P(x)dx} f(x)\,dx + ce^{-\int P(x)dx} \;\middle|\; c \in \mathbb{R} \right\}$$

They *never* express the solution set as a set. They have to get fancy and call it a family, and avoid showing you this, for no good reason, imho. It just clutters up your brain with extra nonsense.

Example 1 Pg 58

$P(x) = -3$

$\frac{dy}{dx} - 3y = 0$ $\quad f(x) = 0$ (Homogeneous EQ'N)

$e^{\int P(x)dx} = e^{-\int 3dx} = e^{-3x} + C = e^{-3x}$, since $C=0$ works.

$\Rightarrow y = e^{-\int P(x)dx} \int e^{-\int P(x)dx} f(x)\,dx + ce^{-\int P(x)dx}$

$= e^{-3x} \int e^{-3x} \cdot 0\,dx + ce^{-3x} = ce^{-3x}$

I showed you this special case, earlier. A method of solution for linear, homogeneous, autonomous ODEs

$y' - 3y = 0$

$(D-3)y = 0$

$D = 3$ OR $y = 0$ (singular)

So $ce^{3x} = y$ is soln. very fast.

$\frac{dy}{dx} - 3y = 0$ is also separable

$\frac{dy}{dx} = 3y$ 3rd way.

$\frac{dy}{3y} = dx$

$\frac{dy}{y} = 3dx$

$\ln|y| = 3x + C$

$|y| = e^{3x+C} = Ke^{3x}$

$\vdots$

$y = Ke^{3x}$

EXAMPLE 2

$\frac{dy}{dx} - 3y = 6$ is also separable

$\frac{dy}{dx} = 3y + 6 = 3(y+2)$

$\int \frac{dy}{y+2} = \int 3\,dx$

$\ln|y+2| = 3x + C$

$\vdots$

$y + 2 = Ke^{3x}$

$y = Ke^{3x} - 2$ Defined $\forall\, x \in \mathbb{R}$

Example 3 **This one seems to call for 2.3 technique. (You may work the previous 2 examples by the 2.3 method, as well. It's just a bit cumbersome.**

$x\frac{dy}{dx} - 4y = x^6 e^x$ **Note $x = 0$ is a singular point**

$\Rightarrow \frac{dy}{dx} - \frac{4}{x}y = x^5 e^x$

$\Rightarrow P(x) = \frac{4}{x}$ & $f(x) = x^5 e^x$

$\Rightarrow \int P(x)\,dx = 4\ln|x|$ $\Rightarrow e^{\int P(x)\,dx} = e^{-4\ln|x|} = e^{-\ln(|x|^4)} = e^{-\ln(x^4)}$

$= e^{\ln(x^{-4})} = \boxed{\frac{1}{x^4} = e^{\int P(x)\,dx}}$ Choose $c=0$ as constant of integration.

$\int e^{\int P(x)dx} f(x)\,dx = \int \frac{1}{x^4}\cdot x^5 e^x\,dx = \int x e^x\,dx$

$\begin{pmatrix} u = x & dv = e^x dx \\ du = dx & v = e^x \end{pmatrix} = uv - \int v\,du = xe^x - \int e^x dx = \boxed{xe^x - e^x + C}$

Don't forget

write sol'n (my way)!

$$y = \frac{\int e^{\int P(x)dx} f(x)\,dx}{e^{\int P(x)dx}} = e^{-\int P(x)dx}\int e^{\int P(x)} f(x)\,dx$$

$$= \frac{xe^x - e^x + C}{\left(\frac{1}{x^4}\right)} = x^4\left(xe^x - e^x + C\right) = \boxed{x^5 e^x - x^4 e^x + C x^4 = y}$$

$\forall\, C \in \mathbb{R}$

This solution holds on $(0,\infty)$ or $(-\infty, 0)$

Example 4 is separable, fwiw.

2.2 way

SINGULAR POINTS : $x=\pm 3$

(zeros of $a_1(x) = x^2-9$)

$$(x^2-9)\frac{dy}{dx} + xy = 0$$

$$(x^2-9)\frac{dy}{dx} = -xy$$

$$\frac{dy}{y} = -\frac{x}{x^2-9} = -\left[\frac{A}{x-3} + \frac{B}{x+3}\right] = -\left[\frac{\frac{1}{2}}{x-3} + \frac{\frac{1}{2}}{x+3}\right]$$

$$\frac{\frac{1}{2}(x+3)+\frac{1}{2}(x-3)}{x^2-9} = \frac{\frac{1}{2}(2x)}{x^2-9} \checkmark$$

$$\frac{A}{x-3} + \frac{B}{x+3} = \frac{x}{x^2-9}$$

$$A(x+3) + B(x-3) = x$$

$x=-3 \Rightarrow$ $-6B=-3$, $B=\frac{1}{2}$

$x=3 \Rightarrow$ $6A=3$, $A=\frac{1}{2}$

$$\therefore \ln|y| = -\frac{1}{2}\left[\ln|x-3| + \ln|x+3| + C\right]$$

$$|y| = e^{-\frac{1}{2}\left[\ln|x-3| + \ln|x+3| + \ln(\hat{c})\right]} = e^{-\frac{1}{2}(\ln(|x-3||x+3|\hat{c}))}$$

$$e^{-\ln\left((|x-3||x+3|\hat{c})^{\frac{1}{2}}\right)} = \frac{1}{\sqrt{\hat{c}}\,|x-3||x+3|} = \frac{K}{|x-3||x+3|}$$

$$y = \frac{K}{\sqrt{|x-3||x+3|}}, \text{ where } K = \pm\frac{1}{\sqrt{\hat{c}}}$$

Now $|x^2-9| = |(x+3)(x-3)| = \begin{cases} (x+3)(x-3) & x \in (-\infty,-3)\cup(3,\infty) \\ -(x+3)(x-3) & x \in (-3,3) \end{cases}$

$$\therefore y = \frac{K}{\sqrt{(x-3)(x+3)}} \quad \text{for some } K.$$

Defined on $(-\infty,-3)$ & $(3,\infty)$

We can also find sol'ns on $(-3,3)$,

but we'd need to play with absolute values.

$$\ln|y| = -\ln\sqrt{c|x-3||x+3|} = \ln\sqrt{c(-(x+3)(x-3))} \quad \text{on } (-3,3)$$

$$= \ln\left(\frac{1}{\sqrt{c|x-3||x+3|}}\right) \Rightarrow$$

$$y = \frac{\frac{1}{\sqrt{c}}}{\sqrt{9-x^2}} = \frac{K}{\sqrt{9-x^2}} \quad \text{on } (-3,3)$$

Wow! That was HARD!

§2.3 way:

$$(x^2-9)\frac{dy}{dx} + xy = 0$$

$$\frac{dy}{dx} + \frac{x}{x^2-9}y = 0 \qquad P(x) = \frac{x}{x^2-9}, \quad f(x) = 0$$

$$\int P(x)\,dx = \int \frac{x}{x^2-9}\,dx = \frac{1}{2}\left[\ln|x-3| + \ln|x+3| + \ln(\hat{c})\right]$$

$$= \frac{1}{2}\ln\left|\hat{c}\,|x^2-9|\right|$$

$$= \ln\left(\sqrt{\hat{c}\,|x^2-9|}\right) \Rightarrow$$

$$e^{\int P(x)dx} = \sqrt{\hat{c}\,|x^2-9|} = K\sqrt{x^2-9} \quad \text{on } (-\infty,-3)\cup(3,\infty)$$

$$\therefore y = e^{-\int P(x)dx}\int e^{\int P(x)dx} f(x)\,dx + ce^{-\int P(x)dx}$$

$$= 0 + \frac{c}{e^{\int P(x)dx}} = \frac{K}{\sqrt{x^2-9}}$$

This is defined on $(-\infty,-3)\cup(3,\infty)$

I'm not too interested in embedding an integration by parts within a differential equations question on the written test.

Look for a separate integration by parts section for things like $\int f(x)\,dx$, where $f(x)$ is...

$x\sin(x)$, $\sin(\sqrt{x})$

$\int \sin(\sqrt{x})\,dx$

$u = \sqrt{x} = x^{\frac{1}{2}}$

$\Rightarrow du = \frac{1}{2}x^{-\frac{1}{2}}dx$

$\Rightarrow dx = 2x^{\frac{1}{2}}du = 2u\,du$!

$\int \sin(\sqrt{x})\,dx = \int \sin(u)(2u\,du) = 2\int u\sin(u)\,du$

reduces to a $\int x\sin(x)\,dx$ situation

https://www.youtube.com/watch?v=zY3HYPzpRec&t=57s