Organizational Matters:

I'm pushing Chapter 2 back a day or three, but when possible, we'll jet through a section. I may do that with some extra video recordings, produced with a bit more (editing out waste-of-time). I think we need a bigger asynchronous component to the learning. That happens when it's being taught the first time, but the cohort is small, and we can make up for a lot of that with more custom-tailored product for a small cohort of 4.

"What do you need? OK. Will do." You just have to be brave enough to tell me!

The Course Schedule (Click here!)

Chapter 2 Written Work due Friday, 9/20.

Course schedule way outta whack.

Section 2.2 - Separable Equations

Recall the 1st-order ODE in normal form:

$$\frac{dy}{dx} = f(x,y)$$

What if $f(x,y) = g(x)$?

Then $\frac{dy}{dx} = g(x) \Rightarrow$

$dy = g(x)\,dx$

$y = y(x)$, so $dy = y'(x)\,dx$

$\Rightarrow \int y'(x)\,dx = \int g(x)\,dx$

$y(x) = \int g(x)\,dx$ Integrate to a solution.

$f(x,y) = \boxed{g(x)h(y) = \frac{dy}{dx}} \rightarrow$ Definition 2.2.1 "Separable Equation.

$\frac{dy}{dx} = g(x)h(y) \Rightarrow$

$$\int \frac{dy}{h(y)} = \int g(x)\,dy$$

Let $p(y) = \frac{1}{h(y)} \Rightarrow$

$$\int p(y)\,dy = \int g(x)\,dx$$

The domain of this eq'n isn't the same as the domain of the original, so our new might have holes in it.

Notice that y such that $h(y) = 0$, then $\frac{dy}{dx} = g(x)h(y)$ has sol'n $y = c$ for any y that makes $h(y) = 0$

This will be a singular solution.

Our technique can't see it.

separable		nonseparable
$\frac{dy}{dx} = y^2xe^{3x+4y}$	and	$\frac{dy}{dx} = y + \sin x$

Always look to WebAssign and D2L for the "hard deadline."

In Problems 1–22 solve the given differential equation by separation of variables.

10. $\frac{dy}{dx} = \left(\frac{2y+3}{4x+5}\right)^2 = \frac{(2y+3)^2}{(4x+5)^2} \Rightarrow$

$$\int \frac{dy}{(2y+3)^2} = \int \frac{dx}{(4x+5)^2}$$

$u = 2y+3$ $\quad u = 4x+5$

$du = 2dy$ $\quad du = 4dx$

$$\Rightarrow \frac{1}{2}\int \frac{2dy}{(2y+3)^2} = \frac{1}{4}\int \frac{4dx}{(4x+5)^2}$$

$$\Rightarrow \frac{1}{2}\int \frac{du}{u^2} = \frac{1}{4}\int \frac{du}{u^2}$$

$$\frac{1}{2}\int u^{-2}du = \frac{1}{4}\int u^{-2}du$$

$$-\frac{1}{2}u^{-1} = -\frac{1}{4}u^{-1} + C$$

$$-\frac{1}{2}(2y+3)^{-1} = -\frac{1}{4}(4x+5)^{-1} + C$$

$$= \frac{1}{2(2y+3)} = \frac{-1}{4(4x+5)} + C$$

want to find y explicitly, but this is an implicit solution.

$$= \frac{-1}{4(4x+5)} + \frac{C(4(4x+5))}{4(4x+5)}$$

My other try, in the wee hours, this morning.

$$= \frac{-1+16Cx+20C}{4(4x+5)} = \frac{1}{2(2y+3)}$$

$$2(2y+3) = \frac{16x+60}{16Cx+20C-1}$$

$$2y+3 = \frac{8x+30}{16Cx+20C-1}$$

$$2y = \frac{8x+30}{\sim\sim} - 3\,\frac{(16Cx+20C-1)}{16Cx+20C-1}$$

$$y = \frac{8x+30-48Cx-60C+3}{2(16Cx+20C-1)}$$

$$\Rightarrow y = \frac{1}{2}\left(\frac{2(4x+5)}{4C(4x+5)-1}\right) - \frac{1}{2}\cdot 3 = \frac{(8x+10)-3(16Cx+20C-1)}{2(16Cx+20C-1)}$$

CLOSE (But no cigar!)

Recall Example 7, Section 1.1:

$x^2+y^2=25$ is an implicit solution of

$\frac{dy}{dx} = -\frac{x}{y}$ defined on $(-5,5)$

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$\frac{x^2}{2} + \frac{y^2}{2} = C$$

$$x^2+y^2 = 2C$$

$x^2+y^2=r^2$, where $r=\sqrt{2C}$

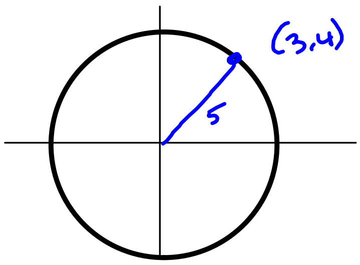
consider IC $y(4)=3$

$y(4)=3 \rightarrow$

$3^2+4^2 = 2C \rightarrow$

$25 = 2C \rightarrow$

$x^2+y^2=25$



Example 3 Initial-Value Problem

Solve the initial-value problem $\dfrac{dy}{dx} = 1 + y^2, \quad y(0) = 0.$

$\Rightarrow$ I.C.

$\Rightarrow \int \frac{dy}{1+y^2} = \int dx$

$\Rightarrow \arctan(y) = x + C$

$\tan(\quad) = \tan(\quad)$

$\Rightarrow \quad y = \tan(x+c)$

IC $y(0)=0$ says

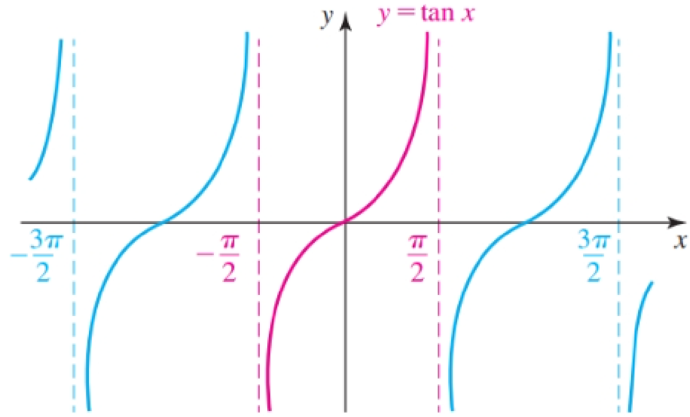
$0 = \tan(0+c)$

want $\tan c = 0$, so $c=0$ works.

so does $c = \pi, 2\pi$

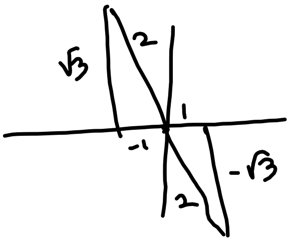
For $c=0$, the domain of the soln is

$x \in (-\frac{\pi}{2}, \frac{\pi}{2})$



Suppose $y(0) = -\sqrt{3}$

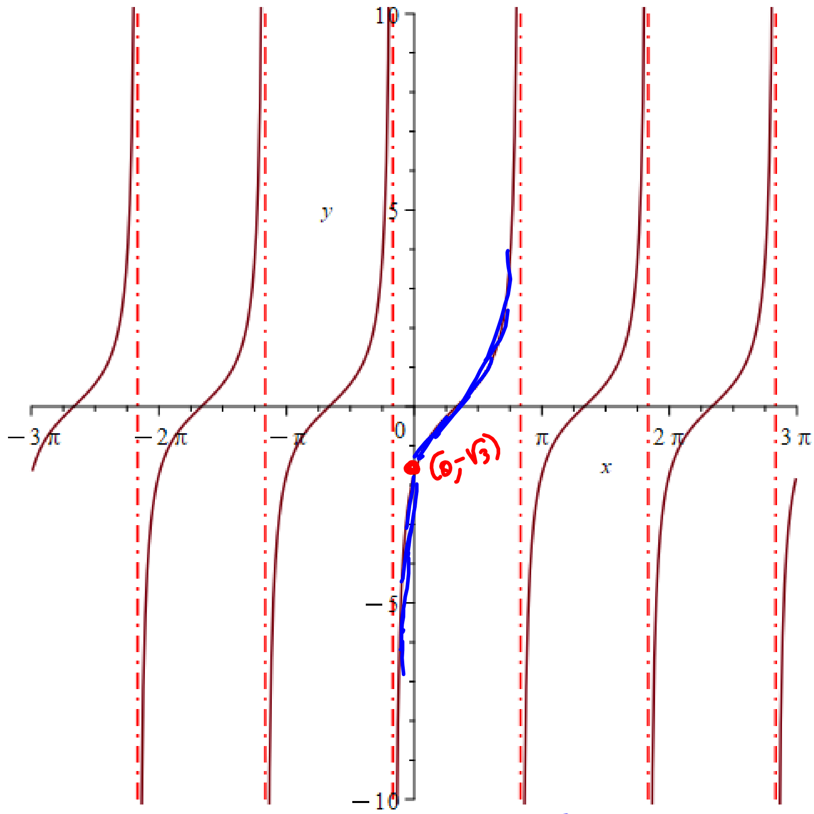
$\Rightarrow \tan(0 + C) = \tan(C) = -\sqrt{3}$



$C = \frac{2\pi}{3}, -\frac{\pi}{3}, \frac{5\pi}{3}$ all work,

$C = -\frac{\pi}{3}$

$y = \tan(x - \frac{\pi}{3})$



$x = -\frac{\pi}{6}$ $x = \frac{5\pi}{6}$

$-\frac{\pi}{2} + \frac{\pi}{3} = \frac{(-3+2)\pi}{6}$

$= -\frac{\pi}{6}$

Domain is $(-\frac{\pi}{6}, \frac{5\pi}{6})$

Recall equations with *extraneous solutions*l

Squaring both sides casts a wide net, but we have to throw back some of the little fishies.

$$\sqrt{x-1}+4=x-3$$
$$\sqrt{x-1}=x-7$$
$$x-1=(x-7)^2=x^2-14x+49$$
$$x^2-15x+50=0$$
$$(x-10)(x-5)=0$$
$$x=10 \text{ OR } x=5$$

$$\sqrt{10-1}+4\stackrel{?}{=}10-3$$
$$\sqrt{9}+4=7$$
$$3+4=7 \checkmark$$
$$\boxed{x=10}$$

$$\sqrt{5-1}+4=5-3$$
$$2+4=2$$
$$6=2 \times$$
$$\boxed{x=5 \text{ is extraneous.}}$$

Something to be on the look out for in the sequel is when our solution technique *loses* solutions, rather than injecting extraneous solutions.

Basically, all we're doing in 2.2 is separating variables and integrating both sides to a solution. It's a big net, but it has *holes* in it!

Recall #10, above.

10. $\frac{dy}{dx} = \left(\frac{2y+3}{4x+5}\right)^2 = \frac{1}{(4x+5)^2} \cdot \underbrace{(2y+3)^2}_{} \overset{\text{SET}}{=} 0 \Rightarrow y = -\frac{3}{2} = r$

is a SINGULAR SOLUTION that is specifically forbidden by the general solution technique.

$$\frac{dy}{(2y+3)^2} = \frac{dx}{(4x+5)^2}$$

$$y(x) = -\frac{48\_C1\,x + 60\_C1 + 8\,x + 7}{2\,(16\_C1\,x + 20\_C1 - 1)}$$ MAPLE SOLUTION

Example 4 Losing a Solution

Solve $\dfrac{dy}{dx} = y^2 - 4.$

$\dfrac{dy}{y^2-4} = dx \Rightarrow$

$\left(\dfrac{1}{y^2-4} = \dfrac{A}{y+2} + \dfrac{B}{y-2}\right)((y-2)(y+2))$

$1 = A(y-2) + B(y+2)$

If this holds @ $y=2$

$1 = A(0) + B(2+2)$

$4B = 1$

$B = \frac{1}{4}$

$y=-2$:

$1 = A(-2-2) = -4A \Rightarrow$

$A = -\frac{1}{4}$

$\displaystyle\int \frac{dy}{y^2-4} = \frac{1}{4}\int \frac{dy}{y-2} - \frac{1}{4}\int \frac{dy}{y+2} = \int dx$

$\Rightarrow \frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = x + C$

$\Rightarrow \ln|y-2| - \ln|y+2| = 4x + 4C$

$\ln\left|\dfrac{y-2}{y+2}\right| = 4x + 4C$

$e^{\ln|\cdots|} = e^{4x+4C} = e^{4x}e^{4C} = Ke^{4x}$ for some K

$\left|\dfrac{y-2}{y+2}\right| = Ke^{4x}$

$\Rightarrow \dfrac{y-2}{y+2} = \pm Ke^{4x} = Me^{4x}$ for $M = \pm K$

$\Rightarrow y - 2 = Me^{4x}(y+2) = yMe^{4x} + 2Me^{4x}$

$y(1-Me^{4x}) = y - yMe^{4x} = 2Me^{4x} + 2$

$\Rightarrow y = \dfrac{2Me^{4x}+2}{1-Me^{4x}}$ is EQN (7) in Book

Note that y = +/- 2 are singular solutions utterly unobtainable by integration, in fact, forbidden by (6).

E5 Recall Integration by Parts

$(e^{2y}-y)\cos(x)\frac{dy}{dx} = e^y \sin(2x) \quad y(0)=0$

$\Rightarrow \frac{e^{2y}-y}{e^y}dy = \frac{\sin(2x)}{\cos(x)}dx = \frac{2\sin(x)\cos(x)}{\cos(x)}dx \longrightarrow$

$\int u\,dv = uv - \int v\,du$

$\int (e^y - ye^{-y})\,dy = \int 2\sin(x)\,dx$

$\int -ye^{-y}du \quad = -\left[uv - \int v\,du\right] = -\left[-ye^{-y} - \int -e^{-y}dy\right]$

$u=y \quad dv = e^{-y}dy$

$du = dy \quad v = -e^{-y}$

$= ye^{-y} + \int e^{-y}dy$

$= ye^{-y} + e^{-y} + C$

So,

$e^y + ye^{-y} + e^{-y} \cancel{+C} = -2\cos(x) + C \quad (8) \leftarrow$ Book

$y = 4!$

$y(0)=0 \longrightarrow$

$e^0 + 0 + e^{-0} = 2 = -2\cos(0) + C = -2 + C$

$\Rightarrow \boxed{4 = C}$

Integral-defined solutions:

Sometimes you can't evaluate the integral on the RHS. Consider:

$$\frac{dy}{dx} = g(x) \implies \int dy = \int g(x)\,dx$$

$$y(x_0) = y_0$$

$$\frac{y'(x)\,dx}{dx} = g(x)$$

$$\int y'(x)\,dx = \int g(x)\,dx$$

$$\int dy = \int g(x)\,dx$$

$$\implies y(x) = \int g(x)\,dx$$

$$\int_{x_0}^{x} y'(t)\,dt = \int_{x_0}^{x} g(t)\,dt$$

$$y(x) - y(x_0) = \int_{x_0}^{x} g(t)\,dt$$

$$\implies y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt$$

Sometimes $\int g(t)\,dt$ has no closed-form antiderivative.

READ 2.3 for Thursday. Linear 1st-order ODE is the cornerstone of all the theory.

The theory is fairly simple. The integrating factor seems like a rabbit out of the hat, but it's really pretty intuitive, once you wrap your head around it. I need you to at least start trying to wrap your head around it, before we launch 2.3.

Heck, take a glance at 2.4, too! Always try to push ahead of where we are, even if you're not going in deep. Just writing down a few things and thinking about them, briefly, really accelerates your understanding on the 2nd pass.

Math is a craft unlike carpentry or metalwork. Math is a craft that is more like writing. "Get something down, even if it's garbage, and you're not even sure what it means!"

At worst, you'll waste a little bit of paper. Any responsible person can plant way more trees than all the paper they use if they write all day, every day, from kindergarten to graduate school.