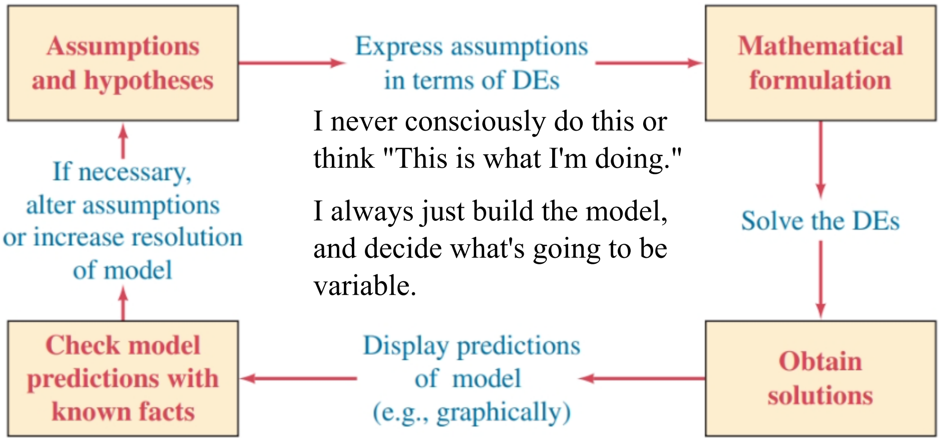
**Section 1.3 - Mathematical Models**

**(i) Level of Resolution** - Start out by treating some or all the variables as constants.

**(ii)** Make assumptions. One we always make is that the phenomenon we 're modeling is somehow continuous and continuously changing (At least twice-differntiable, and therefore continuous and continuous derivative (continuously differentiable at least once).

The physical world is "smooth."

This isn't always a legitimate assumption. Sometimes, if you get "small enough," things start looking "granular." Granulated sugar pours, but it's not liquid. Populations are modeled by exponential functions, but births and deaths occur one at a time.




I never consciously do this or think "This is what I'm doing."

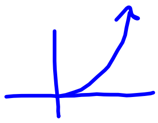
I always just build the model, and decide what's going to be variable.

**Population Dynamics**

Let $P = P(t)$ = population as a function of

$t$ = time (in whatever time units)

Exponential Growth:



$\frac{dP}{dt} \propto P$, i.e.

Rate of change is proportional to the population

$\Rightarrow \frac{dP}{dt} = kP$ for some constant $k \in \mathbb{R}$.

$\Rightarrow \int \frac{dP}{P} = \int k\,dt \Rightarrow$

$\ln|P| = kt + C$

$\Rightarrow e^{\ln(P)} = e^{kt + C}$, as pop. $> 0$

$\Rightarrow P = e^{kt+C} = e^{kt}e^{C} = Ae^{kt}$, where $A = e^C \in \mathbb{R}$

No competition.

No limit to growth.

**Radioactive Decay -** $k < 0$.

**Compound Interest is exponential. The more you have the more you earn, and the earnings are proportional to the amount.**

Note $P(t) = Ae^{kt} \Rightarrow$

$P(0) = P_0 = Ae^0 = A$.

Continuous Compounding

$A(t) = P\left(1 + \frac{r}{m}\right)^{mt}$, $r$ = APR

$m$ = # of periods per year

$t$ = time in years

$i = \frac{r}{m}$ = interest rate per period.

$m = 12$ compounded monthly

$A$ = future Amount $= P\left(1 + \frac{r}{12}\right)^{12t}$

$m \to \infty \Rightarrow \left(1 + \frac{r}{m}\right)^{\frac{m}{r} \cdot rt}$

$= P\boxed{\left(\left(1 + \frac{1}{\frac{m}{r}}\right)^{\frac{m}{r}}\right)}^{rt} \xrightarrow{m \to \infty} Pe^{rt}$ continuous compounding.

$\boxed{\left(1 + \frac{1}{\frac{m}{r}}\right)^{\frac{m}{r}}} \xrightarrow{m \to \infty} e$

$\left(1 + \frac{1}{t}\right)^t \xrightarrow{t \to \infty} e$

**Newton's Law of Cooling:** "The rate of temperature change is proportional to the difference between the temperature of the object and the ambient (surrounding) temperature. Your roast cools more rapidly when it first comes out of the oven...

Let $T_A$ = Ambient Temperature &

$T = T(t)$ = Temp. of my lasagna

Then $\frac{dT}{dt} \propto T - T_A$, i.e., $T(t) = k(T - T_A)$

**Spread of a disease -** It's reasonable to assume that a disease spreads more rapidly as more sick people come into contact with healthy people. We assume that the rate of disease spread is controlled by the number of encounters between sick people and uninfected people.

We assume that the number of these interactions is proportional to the product of the number of people in each. This is a bit of a reach, but not crazy.

**Joint Proportionality assumption.**

Let $x(t)$ = the number of infected people &

$y(t)$ = " " " people as yet unexposed.

$\Rightarrow \frac{dx}{dt} \propto xy$, i.e., $\frac{dx}{dt} = kxy$ for some $k \in \mathbb{R}$

**Bring one sick person into a population of *n* people. Then**

$x + y = n+1$ & so

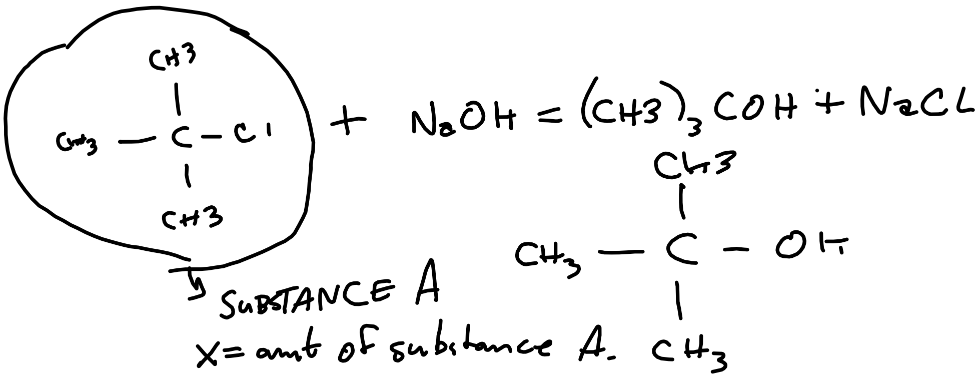
$y = n+1-x \Rightarrow$

$\frac{dx}{dt} = kxy = kx(n+1-x)$, with

i.c. $x(0) = 1$ (Patient zero!)

**Chemical Reactions -** Book plays fast and loose in this model's explanation.

I'm not goig to do a whole lot more with it, other than to sort of explain the chemical reaction in the book.  Just remember "It is natural to assume" doesn't mean "It's absolutely correct to assume."




**The assumption made is that only the concentratio ot the t-butyl chloride controls the rate of reaction, but obviously it depends on the available NaOH in the solution...**

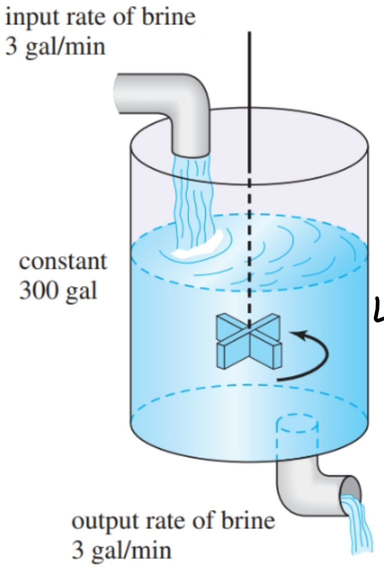
**But forgetting that, it's "reasonable to assume" that the reaction rate is jointly proportional to the amount of remaining, unreacted reagents.**

Then $\frac{dx}{dt} = k(\alpha - x)(\beta - x)$, **Joint Proportionality assumption.**

where $k$ = constant, $x$ = amt of A remaining,
$\alpha$ = amt of A, originally, and $\beta - x$ = amt of B, originally.
(in moles)

This is called a 2nd order reaction, and it's not entirely satisfying.

**Mixtures -** Mixing of two salt mixtures. We're adding a salt mixture to an existing salt mixture, and draining the new mixture at the same rate that the new mixture is being added.




**See #3 in Homework!**

**Concentration of salt in input flow:**

Let $A$ = amt of salt in Tank (in lbs)

$$\frac{dA}{dt} = \left(\begin{matrix}\text{Input Rate} \\ \text{of salt}\end{matrix}\right) - \left(\begin{matrix}\text{Output Rate} \\ \text{of salt}\end{matrix}\right)$$

Let $c_T$ = concentration of salt in the tank

$c_I$ = " " " input pipe.

Let $R_I$ = rate at which salt enters

$R_L$ = " " " " leaves.

Let $R_V$ = " " " mixture is pumped into / out of tank.

Then $\frac{dA}{dt} = R_I - R_L$

$= c_I R_V - c_T R_V$.

Now $C_T$ is going to change over time

Let $V$ = Total Volume of the tank.

Then $c_I R_V - c_T R_V = c_I R_V - \frac{A(t)}{V} R_V = \frac{dA}{dt}$

$$\frac{dA}{dt} = c_I R_V - \frac{A}{V} R_V$$

$$= -\frac{R_V}{V} A + c_I R_I$$

From Book: $R_V = \frac{3\text{ gal}}{\text{min}}$, $V = 300$ gal, $c_I = 2$ lb/gal

$R_I = c_I R_V = \left(\frac{2\text{ lb}}{\text{gal}}\right)\left(\frac{3\text{ gal}}{\text{min}}\right) = \frac{6\text{ lb}}{\text{min}}$

$R_I = \left(\frac{A}{300}\right)\left(\frac{3\text{ gal}}{\text{min}}\right) = \frac{1}{100} A \longrightarrow$

$$\frac{dA}{dt} = 6 - \frac{1}{100} A$$

**Whether this be positive or negative depends on if the incoming mixture is saltier than the mixture already in the tank.**

**Got the last problem on Chapter 1 Solutions done.**

**Getting questions on 1.3 #5. Glad you looked at it, but I'm sad that you were put under unnecessary stress to finish it.**

28. Tractrix A motorboat starts at the origin and moves in the direction of the positive $x$-axis, pulling a waterskier along a curve $C$ called a **tractrix.** See Figure 1.3.23. The waterskier, initially located on the $y$-axis at the point $(0, a)$, is pulled by a rope of constant length $a$ that is kept taut throughout the motion. At time $t > 0$ the waterskier is at point $P(x, y)$. Assume that the rope is always tangent to $C$. Use the concept of slope to determine a differential equation for the path $C$ of motion.

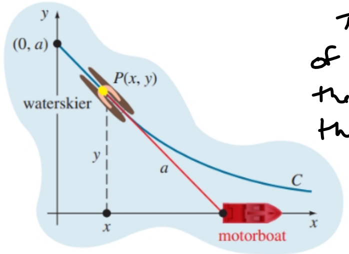



Figure 1.3.23 Waterskier in Problem 28

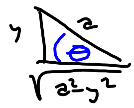
The hook is that the slope of the curve $y = f(x)$ is always the slope of the line formed by the rope.

Introduce a variable $z$ that gives the distance in the x-direction from the back of the boat to the x-position of the skier:

$z = \sqrt{a^2 - y^2(t)}$ by Pythagorus

= horizontal distance from back of boat to skier. As a function of y.

Let $\theta$ = the angle between the rope and the horizontal path of the boat (in diagram).




Then $\tan\theta = \dfrac{y}{\sqrt{a^2-y^2}}$ and

$0 < \theta < \frac{\pi}{2}$

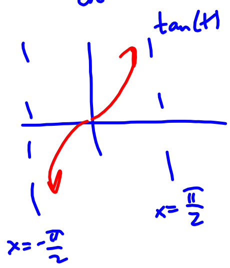
(This is undefined at $t=0$)

$\tan\theta$ will give us the negative of the slope, because the $\Delta y$ is negative

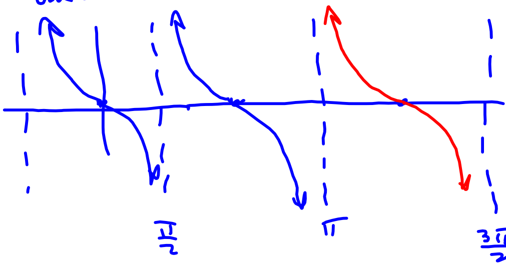
This gives us the negative of the slope of the curve @ the skier's position $(x, y(x))$

$$\tan\theta = \frac{dy}{dx} = -\frac{y}{\sqrt{a^2-y^2}}$$

No "t" in the model. For that, we'd need to know $\frac{dx}{dt}$!

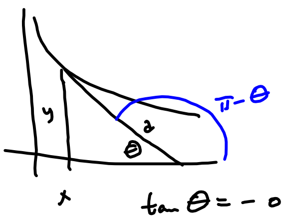



$\tan(\pi - t) = \tan(-(t-\pi))$:



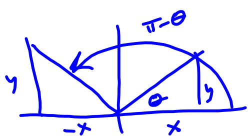

**This was just me trying to make the thing negative by replacing theta with pi-theta, looking at the obtuse angle measured from in front of the boat to the rope.**

$\tan(\theta) = -\tan(\pi - \theta)$



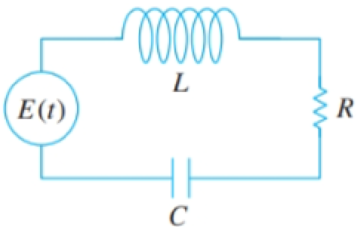

use $\pi-\theta$ & the sign's taken care of

$\tan\theta = -$ of the slope @ x




$\tan\theta = -\tan(\pi - \theta)$

**Series Circuits - Kirchhoff's Laws**




**LRC Circuit**

**Inductance $L$ in henries (h)**

**Current $i(t)$ in amps**

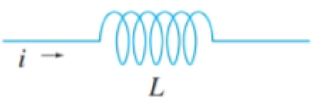
**Resistance $R$**

**Capacitance $C$.**

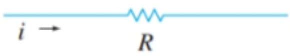
**The charge on the capacitor at time $t$ is $q(t)$**

**By Kirchhoff's 2nd Law, the "impressed" voltage $E(t)$ on a closed loop must equal the sum of the voltage drops in the loop.**

*Inductor*
inductance $L$: henries (h)
voltage drop across: $L\frac{di}{dt}$




*Resistor*
resistance $R$: ohms ($\Omega$)
voltage drop across: $iR$




*Capacitor*
capacitance $C$: farads (f)
voltage drop across: $\frac{1}{C}q$

FACT/DEFINITION: $i(t) = \frac{dq}{dt}$ = change in charge

Inductance

Then $E(t) = L\frac{di}{dt} + iR + \frac{1}{C}q$

$$= L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q$$

#4 on HW