Section 1.1

Definition 1.1.1 Differential Equation

An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a **differential equation (DE).**

$$\frac{dy}{dx} = 0.2xy. \qquad (1)$$

(a) The equations

x is independent

an ODE can contain more than one unknown function

t is independent

$$\frac{dy}{dx} + 5y = e^x, \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \quad \text{and} \quad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y \qquad (2)$$

are examples of ordinary differential equations.

(b) The following equations are partial differential equations:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \qquad (3)$$

***x* and *y* are independent.**

x *and* y *are independent*

***x* and *t* are independent**

Liebniz: $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$; $\frac{d^2y}{dt^2} = -32$

Lagrange: y', y'', y''' ; $y'' = -32$, $y''(t) = -32$

$\ddot{S} = -32$ (Newton)

$$\left.\begin{array}{l} y'' + 3y' + 2y = 0 \\ \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0 \end{array}\right\} \text{2nd-order}$$

$y' = 2$ 1st-order

$y''' = 2$ 3rd-order

Other representations

$M(x,y)\,dx + N(x,y)\,dy = 0$, e.g.:

$(y-x)\,dx + 4x\,dy = 0 \Longrightarrow$

$y - x + 4x\frac{dy}{dx} = 0 \Longrightarrow 4x\frac{dy}{dx} + y = x$

$4xy' + y = x$

An *n*th-order ordinary (one independent variable) differential equation in one dependent variable may be expressed in the following form:

EQN (4) in text

$F(x, y, y', \ldots, y^{(n)}) = 0$, e.g.,

$4xy' + y = x \Longrightarrow 4xy' + y - x = 0 \Longrightarrow$

$F(x, y, y') = 0$, $\overline{n=1}$

can be written as

$y' = f(x,y)$, e.g., $y' = \frac{x-y}{4x} = f(x,y)$

In general, $F(x, y, y', \ldots, y^{(n)}) = 0 \Longrightarrow$

$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$ NORMAL FORM

$\underset{\text{1st ORDER}}{y' = f(x,y)}$ or $\underset{\text{2nd ORDER}}{y'' = f(x,y,y')}$ OR $\underset{\text{5th ORDER}}{y^{(5)} = f(x, y, y', y'', y^{(3)}, y^{(4)})}$

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Linear ODEs: All of the dependent variables in the equation are to the first power, and all the coefficients are functions of the independent variable (including constant coefficients).

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

We'll spend most of our time on linear 1st- and 2nd-order ODEs, basically because they're the easiest to solve!

All other ODEs are nonlinear ODEs.

$$(y - x)\,dx + 4x\,dy = 0, \quad y'' - 2y + y = 0, \quad x^3\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

nonlinear term: coefficient depends on y ↓ | nonlinear term: nonlinear function of y ↓ | nonlinear term: power not 1 ↓

$$(1 - y)y' + 2y = e^x, \quad \frac{d^2y}{dx^2} + \sin y = 0, \quad \text{and} \quad \frac{d^4y}{dx^4} + y^2 = 0$$

Special: A Nonlinear ODE that can be transformed into two nonlinear equations in Normal Form:

$$(y')^2 + 2xy' - y = 0$$

$(y')^2 + 2xy' + x^2 = y + x^2$

$(y'+x)^2 = x^2 + y$

$\sqrt{(y')^2} = \sqrt{x^2+y}$

$|y'| = \sqrt{x^2+y}$

$y' = \pm\sqrt{x^2+y}$ → $y' = \sqrt{x^2+y}$ OR $y = -\sqrt{x^2+y}$ → NOT "AND" (Book flubbed.)

Definition 1.1.2 Solution of an ODE

Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

$F(x, \phi(x), \ldots, \phi^{(n)}(x)) = 0$

"ϕ satisfies the eq'n."

$\boxed{E}$ $y = e^{2x}$ solves

$y' - 2y = 0$

$\boxed{Pf}$ $2e^{2x} - 2e^{2x} = 0$ ✓

$I = (-\infty, \infty)$ = Interval on which this holds.

Since we're talking about functions that solve these equations on AN INTERVAL *I*, be careful about solution functions with holes or gaps in their domain, for example,

$y = \frac{1}{x}$ solves $xy' + y = 0$, since

$x\frac{dy}{dx} = -y$

$x\,dy = -y\,dx$

$\frac{dy}{y} = -\frac{dx}{x} = \frac{dx}{-x}$

$\ln y = -\ln(x) = \ln\left(\frac{1}{x}\right)$

$y = \frac{1}{x}$, which has domain $(-\infty, 0) \cup (0, \infty)$

Note $y=0$ is an explicit sol'n to all of these. We'll be looking for nontrivial sol'ns.

So we have to PICK an interval on which y is differentiable, either $(0, \infty)$ or $(-\infty, 0)$

$$F(x, y, y', \dots, y^{(n)}) = 0, \qquad (4)$$

Definition 1.1.3 Implicit Solution of an ODE

A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ordinary differential equation (4) on an interval I, provided that there exists at least one function ϕ that satisfies the relation as well as the differential equation on I.

Explicit:

$y = 2e^{x} + 3e^{-x}$

Implicit:

$2xy - 5y + x^2y^2 = 0$

In Problems 21 and 22 verify that the indicated expression is an implicit solution of the given first-order differential equation. Find at least one explicit solution $y = \phi(x)$ in each case. Use a graphing utility to obtain the graph of an explicit solution. Give an interval I of definition of each solution ϕ.

21. $\dfrac{dX}{dt} = (X-1)(1-2X)$; $\ln\left(\dfrac{2X-1}{X-1}\right) = t$

$\dfrac{dx}{dt} = (x-1)(1-2x)$

$\ln\left(\dfrac{2x-1}{x-1}\right) = t \implies$

$$\frac{(x-1)}{2x-1} \cdot \frac{\dfrac{(2x')(x-1) - (2x-1)x'}{(x-1)^2}}{\dfrac{(x-1)}{2x-1} \cdot \dfrac{2x-1}{x-1}} = 1$$

Not the best technique.

$$\frac{2x'x - 2x' - 2xx' + x'}{(x-1)(2x-1)} = \frac{x'(2x - 2 - 2x + 1)}{(x-1)(2x-1)} = \frac{-1}{(x-1)(2x-1)}x' = 1$$

$$\implies x' = -(x-1)(2x-1) = (x-1)(1-2x).$$

E7 $x^2+y^2=25$
solves $\frac{dy}{dx}=-\frac{x}{y}$.

Families of Solutions:

Recall:

Explicit:

$y=2e^x+3e^{-x}$

Solves

$y''-y=0$

But so does any function of the form

$y=c_1e^x+c_2e^{-x}$ One way to express this is
$\{c_1e^x+c_2e^{-x} \mid c\in\mathbb{R}\}$

A sol'n of an n^{th} order ODE
$$F(x,y,y',\dots,y^{(n)})=0$$
is an n-parameter family
$$G(x,y,c_1,\dots,c_n)=0$$
where $c_1,c_2,\dots,c_n$ are constants.

$y=c_1e^x+c_2e^{-x}$ is how I'll try to write it,
but formally, $G(x,y,c_1,c_2)=c_1e^x+c_2e^{-x}-y=0$

General Solution - Before the c's are defined.

Particular Solution - Specific values of the constant *c* are selected or found.

E10

$$xy' - 4y = 0$$

$$x\,dy = 4y\,dx$$

$$\frac{dy}{y} = \frac{4\,dx}{x}$$

$$\ln y = +\ln(x) + \ln c$$

$$= \ln(x^4) + \ln c$$

$$= \ln(cx^4)$$

$y = cx^4$ is soln on $(-\infty, \infty)$, even though the soln method says $x > 0, y > 0$. or @ least $y, x \neq 0$.

But we're not bound by that restriction (an artifact of my solution method), when looking at the equation and asking "Is this a solution?" Note that there's yet another solution that works, namely,

$$y = \begin{cases} -x^4 & \text{if } x < 0 \\ x^4 & \text{if } x \geq 0 \end{cases}$$

A single choice of 'c' traps you on one or the other

Singular Solution

A solution that is not a member of the one-parameter family of solutions of the equation

$\frac{dy}{dx} = xy^{\frac{1}{2}}$ (We will solve in C2)

$y = (\frac{1}{4}x^2 + c)^2, c \geq 0$

But $y = 0$ is a sol'n, but not in $\{(\frac{1}{4}x^2+c)^2 \mid c \geq 0\}$.

→ singular sol'n.

System of ODEs:

A system of two first-order differential equations is given by:

$$\frac{dx}{dt} = f(t, x, y)$$

$$\frac{dy}{dt} = g(t, x, y)$$

OR

$$x_1' = f(t, x_1, x_2)$$

$$x_2' = g(t, x_1, x_2)$$

A SOLUTION of such a system is a pair of diff'bl functions $x = \phi_1(t)$, $y = \phi_2(t)$ defined on a common interval I.

Remarks, Pg 11

(i) $M(x,y)dx + N(x,y)dy = 0$ may or may not be linear, depending on which variable is independent.

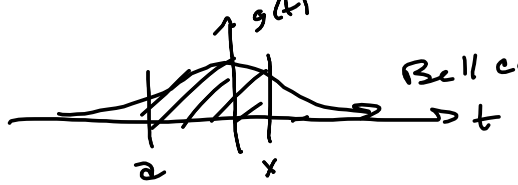
(ii) $f'(x) = g(x) \Rightarrow$

$f(x) = \int_a^x g(t)\,dt + C$ may not be elementary (Not closed-form),

eg. $f'(x) = 3e^{x^2}$, as

$f(x) = \underbrace{\int_a^x 3e^{t^2}dt}_{} + C$

can't evaluate analytically. (But digitally, we can)



Bell curve Area!

Elementary Functions: constant, plynomial, rational, exponential, logarighmic, trigonometric, and inverse trigonometric, plus all algebraic and transcendental functions of these functions.

(iii) Not all DE's have solutions See #51

(v) Can't always solve

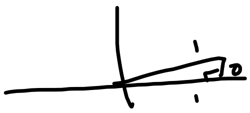
$F(x, y, y', y'', \ldots, y^{(n)}) = 0$ for $y^{(n)}$, that is,

we may not be able to find

NORMAL FORM.

Trig: Solve $\cos(x) - 1 = 0$

$\cos(x) = 1 \implies$



$x = 2n\pi$

$\{x \mid x = 2n\pi,\ n \in \mathbb{Z}\}$

$\{2n\pi \mid n \in \mathbb{Z}\}$

$\mathcal{D}(\sec(x))$

$= \mathcal{D}\left(\frac{1}{\cos(x)}\right) = \{x \mid \cos(x) \neq 0\}$

$= \{x \mid x \neq 2n\pi,\ n \in \mathbb{Z}\}$