4.4 - Method of Undetermined Coefficients

Consider the following differential equation to be solved by the method of undetermined coefficients.

3. $y'' - 9y = (x^2 - 2)\sin(3x)$

Find the complementary function for the differential equation.

$y_p = A\sin(3x) + Bx\sin(3x) + Cx^2\sin(3x)$

$r^2 - 9 = (r-3)(r+3)$

$r = \pm 3$

$y_c = c_1e^{3x} + c_2e^{-3x}$

Find the particular solution for the differential equation.

The RHS $= (x^2-2)\sin(3x)$, so

suppose $y_p = (Ax^2+Bx+C)\sin(3x)$

plug into LHS: $y'' - 9y$:

$$mylhs := diff(diff(y(x), x), x) - 9\cdot y(x)$$

$$mylhs := \frac{d^2}{dx^2}\,y(x) - 9\,y(x)$$

$$g := x \to A\cdot x^2\cdot\sin(3\cdot x) + B\cdot x\cdot\sin(3\cdot x) + C\cdot\sin(3\cdot x)$$

$$g := x \mapsto A\cdot x^2\cdot\sin(3\cdot x) + B\cdot x\cdot\sin(3\cdot x) + C\cdot\sin(3\cdot x)$$

$$subs(y(x) = g(x), mylhs)$$

$$\frac{\partial^2}{\partial x^2}\left(A\,x^2\sin(3\,x) + B\,x\sin(3\,x) + C\sin(3\,x)\right) - 9\,A\,x^2\sin(3\,x) - 9\,B\,x\sin(3\,x) - 9\,C\sin(3\,x)$$

$$simplify(\%)$$

$$\left(-18\,A\,x^2 - 18\,B\,x + 2\,A - 18\,C\right)\sin(3\,x) + 12\cos(3\,x)\left(A\,x + \frac{B}{2}\right)$$

uh-oh!

Looks like we need a cos(3*x*) in there!

Assume $y_p = (Ax^2+Bx+C)\sin(3x) + (Ex^2+Fx+G)\cos(3x)$

Put this in the Differential Equation:

$$g := x \to \left(A\cdot x^2 + B\cdot x + C\right)\cdot\sin(3\cdot x) + \left(E\cdot x^2 + F\cdot x + G\right)\cdot\cos(3\cdot x)$$

$$g := x \mapsto \left(A\cdot x^2 + B\cdot x + C\right)\cdot\sin(3\cdot x) + \left(E\cdot x^2 + F\cdot x + G\right)\cdot\cos(3\cdot x)$$

$$subs(y(x) = g(x), mylhs)$$

$$\frac{\partial^2}{\partial x^2}\left(\sin(3\,x)\left(A\,x^2 + B\,x + C\right) + \left(E\,x^2 + F\,x + G\right)\cos(3\,x)\right) - 9\sin(3\,x)\left(A\,x^2 + B\,x + C\right) - 9\left(E\,x^2 + F\,x + G\right)\cos(3\,x)$$

$$simplify(\%)$$

$$\left(-18\,E\,x^2 + (12\,A - 18\,F)\,x + 6\,B + 2\,E - 18\,G\right)\cos(3\,x) - 18\left(A\,x^2 + \left(B + \frac{2\,E}{3}\right)x - \frac{A}{9} + C + \frac{F}{3}\right)\sin(3\,x)$$

$$\% - f(x)$$

$$\left(-18\,E\,x^2 + (12\,A - 18\,F)\,x + 6\,B + 2\,E - 18\,G\right)\cos(3\,x) - 18\left(A\,x^2 + \left(B + \frac{2\,E}{3}\right)x - \frac{A}{9} + C + \frac{F}{3}\right)\sin(3\,x) - x^2\sin(3\,x) + 2\sin(3\,x)$$

$simplify(\%)$ WANT $= 0$

$$\left(\left(-18\,A - 1\right)x^2 + \left(-18\,B - 12\,E\right)x + 2\,A - 18\,C - 6\,F + 2\right)\sin(3\,x) + 12\left(-\frac{3\,E\,x^2}{2} + \left(A - \frac{3\,F}{2}\right)x + \underbrace{\frac{B}{2} + \frac{E}{6} - \frac{3\,G}{2}}\right)\cos(3\,x)$$

$-18A = 1$

$\boxed{A = -\frac{1}{18}}$

$-18B - 12E = 0$

$-18B = 0$

$\boxed{B = 0}$

$2A - 18C - 6F + 2 = 0$

$-\frac{2}{18} - 18C - 6\left(-\frac{1}{27}\right) = -2$

$-\frac{1}{9} + \frac{2}{9} - 18C = -2$

$\frac{1}{9} - 18C = -2$

$-18C = -\frac{18}{9} - \frac{1}{9} = -\frac{19}{9}$

$\boxed{C = +\frac{19}{162}}$

$\boxed{E = 0}$

$A - \frac{3F}{2} = 0$

$A = \frac{3F}{2}$

$F = \frac{2}{3}A = \frac{2}{3}\left(-\frac{1}{18}\right)$

$\boxed{F = -\frac{1}{27}}$

$\frac{B}{2} - \frac{3G}{2} = 0$

$B = 3G$

$\boxed{G = 0}$

$: 0$

$$-\frac{1}{18}x^2\sin(3x) + \frac{19}{162}\sin(3x) - \frac{1}{27}x\cos(3x)$$

$y = y_c + y_p = c_1e^{3x} + c_2e^{-3x} + \left(-\frac{1}{18}x^2 + \frac{19}{162}\right)\sin(3x) - \frac{1}{27}x\cos(3x)$

$$C_1e^{3x} + C_2e^{-3x} - \frac{1}{18}x^2\sin(3x) + \frac{19}{162}\sin(3x) - \frac{1}{27}x\cos(3x)$$