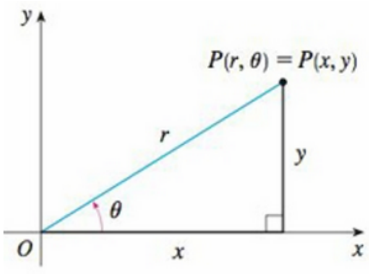
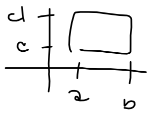
15.4 Double Integrals in Polar Coordinates.



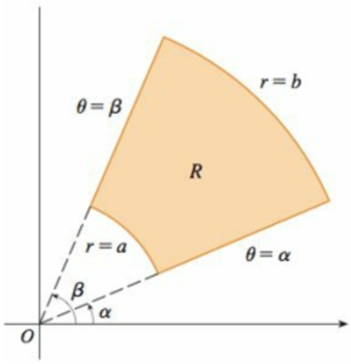
Recall

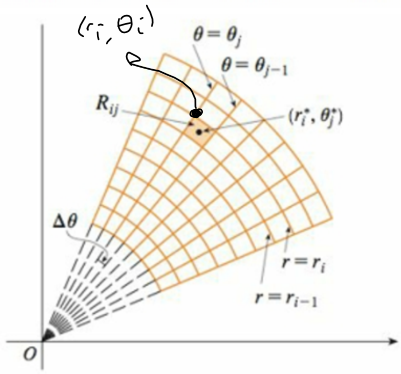
$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$



"Rectangular" Rectangle: $R = \{ (x,y) \mid a \le x \le b,\ c \le y \le d \} = [a,b] \times [c,d]$

polar rectangle $R = \{(r, \theta) \mid a \leqslant r \leqslant b,\ \alpha \leqslant \theta \leqslant \beta\}$

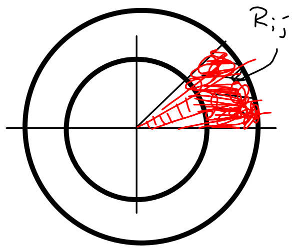




The "center" of the polar rectangle $R_{ij} = \{(r, \theta) \mid r_{i-1} \leq r \leq r_i, \theta_{j-1} \leq \theta \leq \theta_j\}$

has coordinates $r_i^* = \frac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$

Recall: The area of a sector of a circle $A = \frac{1}{2}r^2\theta$



Area of $R_{ij} = \Delta A_{ij} = \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta$

$$= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^*\,\Delta r\,\Delta\theta$$

r_i^* $\Delta r, \Delta\theta \to 0 \Rightarrow r\,dr\,d\theta$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\,\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\,r_i^*\,\Delta r\,\Delta\theta$$

Define $g(r, \theta) = rf(r\cos\theta, r\sin\theta)$. Then the above Riemann Sum becomes:

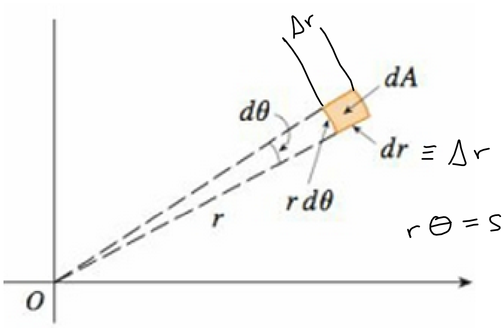
$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\,\Delta r\,\Delta\theta$$

One hopes and expects that in the limit as m, n approach infinity, we obtain a double integral:

2 CHANGE TO POLAR COORDINATES IN A DOUBLE INTEGRAL If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x, y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

A nice way of remembering the area increment dA in polar coordinates is by thinking of the polar rectangle as an ordinary rectangle:



$r\theta = s =$ arc length

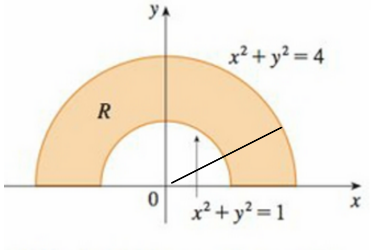
And the smaller we get, the more the arcs involved ARE like straight line segments and the more "parallel" the sides are.

Personally, I just think it's a joke. Hardee-Har! ("r dr")

Why fool with this?

Because some regions are more circular than others, and polar coordinates give us an efficient way to represent things in some cases.

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



(b) $R=\{(r,\theta)\mid 1\le r\le 2,\ 0\le\theta\le\pi\}$

R is a polar rectangle.

Power-reducing formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

R is a polar rectangle, so...

$$\int_0^\pi\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^\pi\int_1^2 3r\cos\theta\, r\, dr\, d\theta + 4\int_0^\pi\int_1^2 r^2\left(\tfrac{1}{2}(1-\cos(2\theta))\right) r\, dr\, d\theta$$

$$= \int_0^\pi\int_1^2 3r^2\cos\theta\, dr\, d\theta + \frac{4}{2}\int_0^\pi\int_1^2 r^3\,(1-\cos(2\theta))\, r\, dr\, d\theta$$

$$= \int_0^\pi \Big[r^3\cos\theta\Big]_{r=1}^{r=2} d\theta + 2\int_0^\pi \left[\frac{r^4}{4}(1-\cos(2\theta))\right]_{r=1}^{2} d\theta$$

$$= \int_0^\pi (2^3-1^3)\cos\theta\, d\theta + \frac{1}{2}\int_0^\pi (2^4-1^4)(1-\cos(2\theta))\, d\theta$$

$$= \int_0^\pi 7\cos\theta\, d\theta + \frac{15}{2}\int_0^\pi (1-\cos(2\theta))\, d\theta$$

$$= \Big[7\sin\theta\Big]_0^\pi + \frac{15}{2}\left[\int_0^\pi d\theta - \frac{1}{2}\int_0^\pi 2\cos(2\theta)\, d\theta\right] = 7(\sin(\pi)-\sin(0)) + \left[\frac{15}{2}\theta\right]_0^\pi - \frac{1}{2}\Big[(-\sin(2\theta))\Big]_0^\pi$$

$$= 0 + \frac{15\pi}{2} + \frac{1}{2}\Big[\sin(2\pi)-\sin(0)\Big] = \boxed{\frac{15\pi}{2}}$$

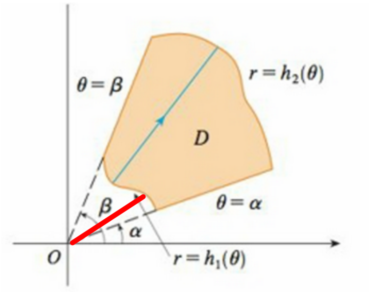


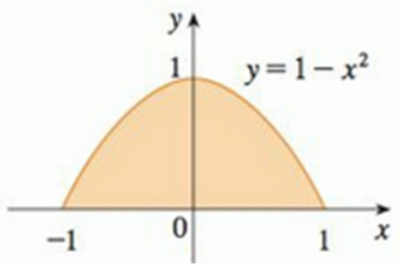
FIGURE 7
$D=\{(r,\theta)\mid \alpha\leqslant\theta\leqslant\beta,\ h_1(\theta)\leqslant r\leqslant h_2(\theta)\}$

3 If f is continuous on a polar region of the form

$$D=\{(r,\theta)\mid \alpha\leqslant\theta\leqslant\beta,\ h_1(\theta)\leqslant r\leqslant h_2(\theta)\}$$

then
$$\iint_D f(x,y)\,dA=\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

2.



11. $\iint_D e^{-x^2-y^2}\,dA$, where D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis

15–18 Use a double integral to find the area of the region.

15. One loop of the rose $r = \cos 3\theta$

36. (a) We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dy\, dx$$
$$= \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}\, dA$$

where D_a is the disk with radius a and center the origin. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dA = \pi$$