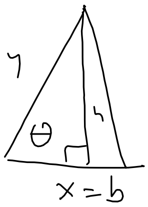
§14.5

43. One side of a triangle is increasing at a rate of 3 cm/s and a second side is decreasing at a rate of 2 cm/s. If the area of the triangle remains constant, at what rate does the angle between the sides change when the first side is 20 cm long, the second side is 30 cm, and the angle is $\pi/6$?

Want $\frac{d\theta}{dt}$ $\Big|_{(x,y,\theta)=(30,20,\frac{\pi}{6})}$

$\frac{dy}{dt} = \frac{3\text{cm}}{s}$

$\frac{dx}{dt} = -\frac{2\text{cm}}{s}$



$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}xy\sin\theta$$

$$\frac{h}{y} = \sin\theta$$

$$h = y\sin\theta$$

$$\frac{dA}{dt} = \frac{1}{2}\left[\frac{\partial A}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial A}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial A}{\partial\theta}\cdot\frac{d\theta}{dt}\right] = 0$$

§14.6 #s 4, 8, 9, 16, 19, 24, 34, 40, 52

§14.7 #s 3, 6, 11, 19, 21, 26, 30, 33, 39, 40

§14.8 #s 2, 3, 6, 10, 13, 14, 27, 28, 41, 45, 46

45–46 Use a computer to graph the surface, the tangent plane, and the normal line on the same screen. Choose the domain carefully so that you avoid extraneous vertical planes. Choose the viewpoint so that you get a good view of all three objects.

45. $xy + yz + zx = 3$, $(1, 1, 1)$

46. $xyz = 6$, $(1, 2, 3)$

47. If $f(x, y) = xy$, find the gradient vector $\nabla f(3, 2)$ and use it to find the tangent line to the level curve $f(x, y) = 6$ at the point $(3, 2)$. Sketch the level curve, the tangent line, and the gradient vector.

48. If $g(x, y) = x^2 + y^2 - 4x$, find the gradient vector $\nabla g(1, 2)$ and use it to find the tangent line to the level curve $g(x, y) = 1$ at the point $(1, 2)$. Sketch the level curve, the tangent line, and the gradient vector.

49. Show that the equation of the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$$

50. Find the equation of the tangent plane to the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ at (x_0, y_0, z_0) and express it in a form similar to the one in Exercise 49.

51. Show that the equation of the tangent plane to the elliptic paraboloid $z/c = x^2/a^2 + y^2/b^2$ at the point (x_0, y_0, z_0) can be written as

$$\frac{2xx_0}{a^2} + \frac{2yy_0}{b^2} = \frac{z + z_0}{c}$$

52. At what point on the paraboloid $y = x^2 + z^2$ is the tangent plane parallel to the plane $x + 2y + 3z = 1$?

53. Are there any points on the hyperboloid $x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$?

54. Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$. (This means that they have a common tangent plane at the point.)

55. Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.

56. Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the center of the sphere.

57. Show that the sum of the x-, y-, and z-intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

58. Show that the pyramids cut off from the first octant by any tangent planes to the surface $xyz = 1$ at points in the first octant must all have the same volume.

59. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.

60. (a) The plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 1)$.

(b) Graph the cylinder, the plane, and the tangent line on the same screen.

61. (a) Two surfaces are called **orthogonal** at a point of intersection if their normal lines are perpendicular at that point. Show that surfaces with equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ are orthogonal at a point P where $\nabla F \neq \mathbf{0}$ and $\nabla G \neq \mathbf{0}$ if and only if

$$F_xG_x + F_yG_y + F_zG_z = 0 \quad \text{at } P$$

(b) Use part (a) to show that the surfaces $z^2 = x^2 + y^2$ and $x^2 + y^2 + z^2 = r^2$ are orthogonal at every point of intersection. Can you see why this is true without using calculus?

62. (a) Show that the function $f(x, y) = \sqrt[3]{xy}$ is continuous and the partial derivatives f_x and f_y exist at the origin but the directional derivatives in all other directions do not exist.

(b) Graph f near the origin and comment on how the graph confirms part (a).

63. Suppose that the directional derivatives of $f(x, y)$ are known at a given point in two nonparallel directions given by unit vectors $\mathbf{u}$ and $\mathbf{v}$. Is it possible to find ∇f at this point? If so, how would you do it?

64. Show that if $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = \langle x_0, y_0 \rangle$, then

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0$$

[*Hint:* Use Definition 15.4.7 directly.]

$D_{\bar{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b$, where

$\bar{u} = \langle a,b \rangle$

$D_{\bar{u}} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a,b \rangle$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_1 + x_2 + 5x_3 + 7x_4 + \cdots 11x_{10}$

$x_1 + 2x_2 = 5$

$3x_1 + 4x_2 = 6$

$\underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} 1x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$A\bar{x} = \bar{b}$

$A^{-1}(A\bar{x}) = A^{-1}\bar{b}$

$(A^{-1}A)\bar{x} = A^{-1}\bar{b}$

$I\bar{x} = \boxed{\bar{x} = A^{-1}\bar{b}}$

Is all the same, regardless of dimension

Symbolic Sol'n always works the same

COMPUTERS

8 DEFINITION If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \underline{\langle f_x(x, y), f_y(x, y) \rangle} = \boxed{\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}}$$

capital del. ∂ = lower-case del.

This gives us the following notation:

$$\boxed{9} \quad D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Can't visualize this unless you're a mutant, but this is basically saying that we can bump the dimension up to any positive finite integer and the same notation applies.

10 DEFINITION The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

PLUS, using VECTOR NOTATION, we see things collapse to our original 1-Dimensional characterizations (almost).

$$\boxed{11} \quad D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

15 THEOREM Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

The basic idea underlying the proof is:

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$$

And it's maximal when the angle theta is zero!

$$\cos\theta = \frac{\bar{u} \cdot \bar{v}}{\|\bar{u}\|\|\bar{v}\|}$$

$$\bar{u} \cdot \bar{v} =$$

So all these ideas collapse down to some very elegant calculations.