Section 14.3 Partial Derivatives

§14.3 #s 4, 10, 11, 13, 15, 18, 21, 26, 29, 30, 47, 50, 52, 53, 56, 59, 71*, 78, 81, 83

*Clairaut.

$f_x(a,b) = f_x(a,b) = g'(a) \quad \text{where} \quad g(x) = f(x,b)$

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

4 If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notations for Partial Derivatives If $z = f(x, y)$, we write

$D_x f$

∂ = "del"

$z = f(x, y)$

$$f_x = f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y = f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

$$f(x,y) = x^2 \sin(xy) + x^4 y^5 \qquad \text{Find } f_x(x,y),\ f_y(x,y)$$

$f_x = 2x \sin(xy) + x^2 \cos(xy) \cdot y + 3x^3 y^5$

$f_y = x^2 \cos(xy) \cdot x + 5x^4 y^4$

$$\frac{\partial}{\partial x}[xy] = y$$

$$\frac{d}{dx}[7x] = 7$$

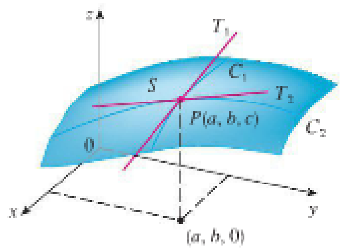


FIGURE 1
The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

EXAMPLE 2 If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

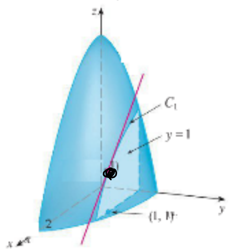
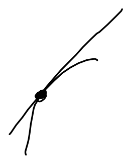


FIGURE 2



f_x vertical plane $\parallel$ to xz-plane

f_y vertical plane $\parallel$ to yz-plane.

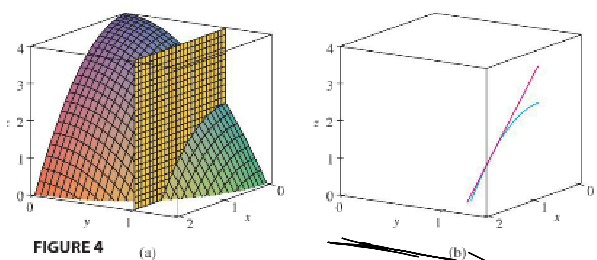


FIGURE 4 (a)

Functions of 3 or more variables...

$f(x_1, x_2, \ldots, x_n)$

f_{x_3}, etc. Same deal.

Higher Derivatives

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial y\, \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial x\, \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

f_{xy} means $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$\frac{\partial^2 f}{\partial y \partial x}$ means the same thing.

(order is reversed from f_{xy} notation

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2}{\partial y \partial x}\left[f\right] = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$$

$$f(x,y) = x^2\sin(xy) + x^4y^5$$

Find $f_x(x,y)$, $f_y(x,y)$, $f_{xx}(x,y)$, $f_{yy}(x,y)$, $f_{xy}(x,y)$, $f_{yx}(x,y)$

$f_x = 2x\sin(xy) + x^2(\cos(xy))\cdot y + 4x^3y^5 = 2x\sin(xy) + \boxed{x^2y\cos(xy)} + 4x^3y^5$

$f_{xx} = 2\sin(xy) + 2x(-\cos(xy))\cdot y + 12xy^5$

$f_{xy} = \frac{\partial^2}{\partial y \partial x}[f] = 2x(\cos(xy))\cdot x + 20x^3y^4 = 2\sin(xy) + 2x(-\cos(xy))y + 2xy\cos(xy) + x^2y(-\sin(xy))\cdot y$

$f_y = x^2(\cos(xy))\cdot x + 5x^4y^4 = x^3\cos(xy) + 5x^4y^4$

$f_{yy} =$

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Smoothness is why.

Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ **harmonic functions;**
heat conduction, fluid flow, and electric potential.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

13 $f(x,y) = x^2y^3$

$$\boxed{\begin{aligned} f_x &= 2xy^3 \\ f_y &= 3x^2y^2 \end{aligned}}$$

$f_{xy} = 6xy^2$

$f_{yx} = 6xy^2$

29 $F(x,y) = \int_y^x \cos(e^t)\,dt = -\int_x^y \cos(e^t)\,dt$

$F_x = \cos(e^x)$

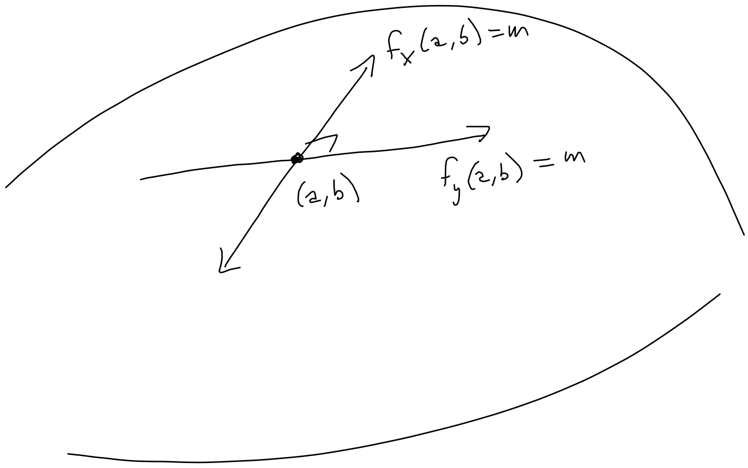
$F_y = -\cos(e^y)$

Fundamental Theorem of Calculus I

$$\frac{d}{dx}\int_0^{x^3} \frac{\cos(\pi t)(t^2-5)}{t^2+42}\,dt$$

Chain Rule for FTC I.

$$= \left(\frac{\cos(\pi x^3)(x^6-5)}{x^6+42}\right)(3x^2)$$



f_x : y is held constant.

f_x is the slope in the plane $y=b$

& it's an (x,z) thing

f_y is a (y,z) thing

Suppose $f_x(2,4)=7$ $\quad z=7(x-2)=7x-14$

$x=x$, $y=4$ $\quad x=t,\ y=4,\ z=7t-14$

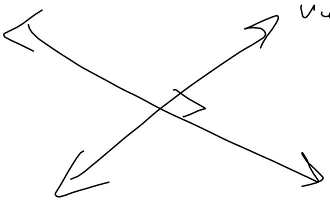
$f_y(2,4)=-3$

Parametric eq'n $\quad z=-3(y-4)=-3y+12$

$x=2,\ y=t,\ z=-3t+12$

The big idea I'm aiming at

§14.4



vector in tangent plane @ $(2,4)$ f_x direction

Vector in the tangent plane @ $(2,4)$ in f_y direction

Find vectors corresponding to f_x, f_y

Then take their cross product for $\vec{n}$.

This gives tangent plane for a surface.

$f(x,y)=x^2y^3$ @ $(1,2,8)$ Find tangent plane.

$f_x = 2xy^3 \Rightarrow f_x(1,2) = 16$

Tan line:

$z = 16(x-1)+8 \ , \ y = 2$

$= 16x - 16 + 8$

$= 16x - 8, \ x = x, \ y = 2$

$x = t, \ y = 2, \ z = 16t - 8$

$\langle 0, 2, -8\rangle + t\langle 1, 0, 16\rangle$

$f_y = 3x^2y^2$

$f_y(1,2) = 12$

$z = 12(y-2) + 8$

$= 12y - 24 + 8$

$x = 1, \ y = t, \ z = 12t - 16$

$\langle 1, 0, -16\rangle + t\langle 0, 1, 12\rangle$

Take the cross product for $\bar{n}$.

$$\begin{array}{r} \langle 1, 0, 16\rangle \\ \times \ \langle 0, 1, 12\rangle \\ \hline \langle -16, -12, 1\rangle = \bar{n} \end{array}$$

$-16(x-1) - 12(y-2) + 1(z-8) = 0$

$\bar{u} = \langle x-1, y-2, z-8\rangle$ is vector in the plane, provided (x,y,z) is a point on the plane