Section 14.5 The Chain Rule

14.5 #s 1, 4, 7, 10, 13, (17-20 (optional)), 24, 27, 32, 35, 43*, 45*

2 The Chain Rule (Case 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

Curve is z.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Note the interplay of $d - vs - \partial$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$z = x^2y^3 \Longrightarrow$ $x(t) = t^2-1$, $y(t) = \sin(t)$

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$

$$= 2xy^3 \cdot 2t + 3x^2y^2 \cdot \cos t$$

$$= 2(t^2-1)(\sin^3(t))(2t) + 3(t^2-1)^2(\sin^2(t))(\cos(t))$$

3 The Chain Rule (Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Note that we're no longer talking about a space curve, but a surface, now. When x and y both depended on one (the same) parameter t, the function was a 1-dimensional object embedded in 3-space. When they each depend on 2 parameters, you get a 2-dimensional object embedded in 3-space (a surface).

In the 2-parameter case, we say that s and t are **independent** variables, $\mathbb{R}^3$

x and y are **intermediate** variables, and z is the **dependent** variable.

This upgrades to arbitrary number of intermediate and independent variables in the natural way:

4 The Chain Rule (General Version) Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

EXAMPLE 5 If $u = x^4 y + y^2 z^3$, where $x = rse^t$, $y = rs^2 e^{-t}$, and $z = r^2 s \sin t$, find the value of $\cancel{\partial u/\partial s}$ when $\cancel{r = 2, s = 1, t = 0}$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$= (4x^3 y)(se^t) + (x^4 + 2yz^3)(s^2 e^{-t}) + (3y^2 z^2)(2rs\sin(t))$$

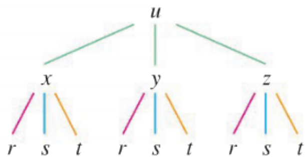


FIGURE 4

EXAMPLE 7 If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find (a) $\partial z/\partial r$ and (b) $\partial^2 z/\partial r^2$.

(a) $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$

$= \frac{\partial z}{\partial x}\cdot 2r + \frac{\partial z}{\partial y}\cdot 2s = 2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}$

(b) $\frac{\partial}{\partial r}\left[\frac{\partial z}{\partial r}\right] = \frac{\partial}{\partial r}\left[2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right]$

$(fg)' = f'g + fg'$

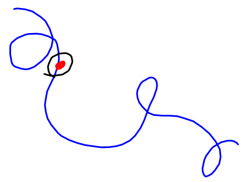
$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$= 2\frac{\partial z}{\partial x} + (2r)\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$

$= 2\frac{\partial z}{\partial x} + 2r\frac{\partial^2 z}{\partial r\partial x} + 2s\frac{\partial^2 z}{\partial r\partial y} = \frac{\partial^2 z}{\partial r^2}$

Chain Rule

Suppose $F(x, y) = 0$ and assume that y is (at least locally) a function of x.



Differentiating both sides w.r.t. x gives $\dfrac{\partial F}{\partial x}\dfrac{dx}{dx} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dx} = 0$

$\dfrac{dx}{dx} = 1$

$$-\frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial F}{\partial x} \Longrightarrow \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

and we obtain a slick formula for $\dfrac{dy}{dx} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\dfrac{F_x}{F_y}$ (Cheat sheet material)

At first, this seems a bit odd way to do things, but it deepens our understanding of some of our techniques for implicit differentiation, and shortens up a lot of the repetitive work involved in using implicit differentiation to say things about curves that are *not* functions.

EXAMPLE 8 Find y' if $x^3 + y^3 = 6xy$.

"OLD"

$3x^2 + 3y^2y' = 6y + 6xy'$

$3y^2y' - 6xy' = 6y - 3x^2$

$y' = \dfrac{6y - 3x^2}{3y^2 - 6x}$

"NEW"

$x^3 + y^3 - 6xy = 0$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{3x^2 - 6y}{3y^2 - 6x}$

Quicker. More formulaic.

$$F(x, y, z) = 0$$

Assume $z = f(x, y)$

$\Longrightarrow$ $F(x, y, f(x, y)) = 0$

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$\Rightarrow$ Assuming x & y are independent of one another, so $\frac{dy}{dx} = 0$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \longrightarrow$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$