Shorthand: $\ni$ means "such that" or "so that"

$\exists$ There is, there exists

$\forall$ For all, for each, for every

$x \in S$ x is in the set S, x is an element of S

$A \Rightarrow B$ A implies B

$A \Leftrightarrow B$, A iff B: A implies B and B implies A, if and only if, A is necessary and sufficient to B

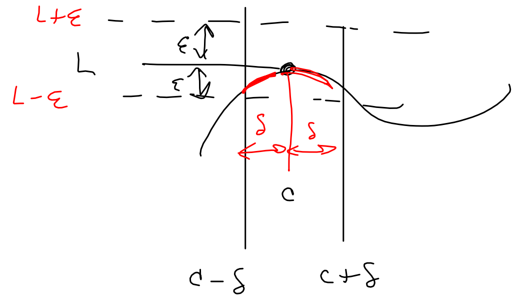
$3x = 2$ iff $x = \frac{2}{3}$

$A \Longleftrightarrow B$

Recall the definition of limit from Calculus I:

$\lim_{x \to c} f(x) = L$ means

$\forall \varepsilon > 0, \exists \delta \ni \forall x$ with $|x - c| < \delta$, we have $|f(x) - L| < \varepsilon$.



$\lim_{x \to 5} 7x + 9 = 44$

Proof

Let $\varepsilon > 0$ be given. Define $\delta = \frac{\varepsilon}{7}$. Then

$0 < |x-5| < \delta \Longrightarrow |7x + 9 - 44| = |7x - 35|$

$= 7|x-5| < 7\delta = 7 \cdot \frac{\varepsilon}{7} = \varepsilon$. $\blacksquare$

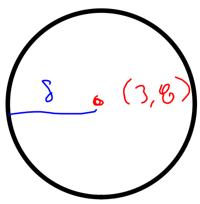
Now we bump it up in dimension!

$|x-5| < \delta$ becomes something like

$(x, y) \longmapsto (3, 8)$

δ is now a distance in the plane dood:

$\sqrt{(x-3)^2 + (y-8)^2} < \delta$



1 Definition Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \varepsilon$

$\lim_{(x,y) \to (a,b)} f(x,y)$ means

$\forall \ \varepsilon > 0, \exists \ \delta > 0 \ni 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$

$\Longrightarrow |f(x,y) - L| < \varepsilon.$

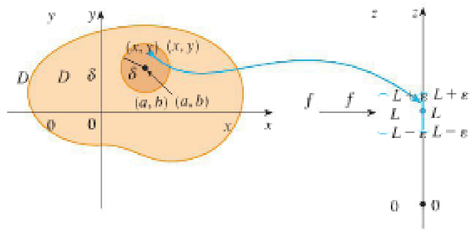


FIGURE 1

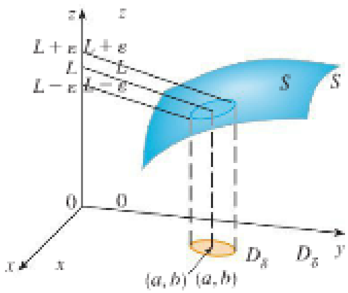


FIGURE 2

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

When the limit does NOT exist, sometimes the only practical way to show it is to be clever in how you make the approach to the limiting input value from a direction (or along a curve) where you get two different results, proving the limit doesn't exist.

Standard "trick."

EXAMPLE 1 Show that $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

$(x, y) = (x, 0) \longrightarrow (0, 0)$

$\dfrac{x^2}{x^2} = 1 \xrightarrow{(x,y)\to(0,0)} 1$ Along the x-axis

$\dfrac{-y^2}{y^2} = -1 \xrightarrow{(x,y)\to(0,0)} -1$ Along the y-axis

$-1 \neq 1$ limit $\nexists$.

other tricks

x Let $x =$ $\boxed{\text{the amount of 22\% alcohol mix, in gallons.}}$

$.22x + .83(50) = .5(x+50)$

a rotating line on
e 6 shows differ-
gin from different

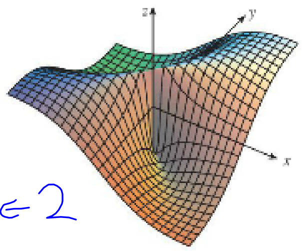


FIGURE 6
$f(x, y) = \frac{xy}{x^2 + y^2}$

EXAMPLE 2

Maybe y'all can help me grok the enrichment tool:

https://www.cengage.com/math/discipline_content/stewartcalc7/2008/14_cengage_tec/publish/deployments/transcendentals_7e/7e_m12_6a.html

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Practical:

$\lim_{(x,y) \to (a,b)}$

$(x, b) \to (a, b)$

$(a, y) \to (a, b)$

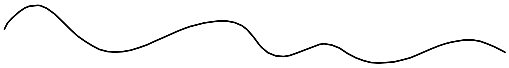
$y = x \qquad y = x$

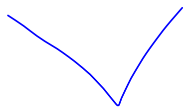
$y = mx$ any line thru $(0,0)$

$(x, y) \to (0, 0)$

Continuity

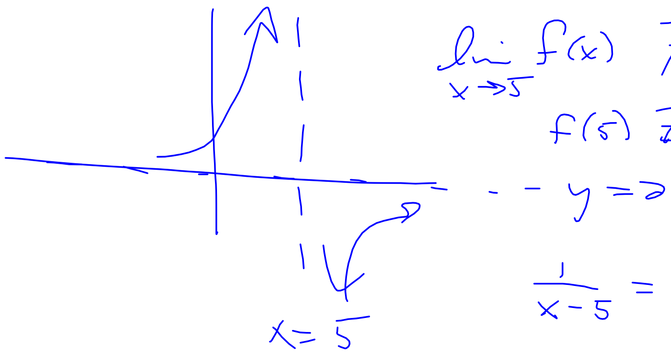
$$\boxed{\lim_{x \to c} f(x) = f(c)}$$





Non example

$$\lim_{x \to 5} f(x) = 7 \neq 5 = f(5)$$



$\lim_{x \to 5} f(x) \nexists$

$f(5) \nexists$ Done.

$$\frac{1}{x-5} = f(x)$$

Now $\lim_{(x,y) \to (a,b)} f(x) = f(a,b)$

§14.2 #s 5, 6, 9, 11, 18, 29, 31, 32, 37.

More practical:

If $f(x,y)$ is continuous @ (a,b), then

$$\lim_{(x,y)\to(a,b)} f(x) = f(a,b)$$

So look for discontinuities for problem areas.

(Look for DOMAIN!)

$$f(x,y) = \frac{x^2-y^2}{x^2+y^2}$$

$x=y=0$ is only problem spot

$$\mathcal{D} = \mathbb{R}^2 \setminus \{(0,0)\}$$

EXAMPLE 4 Find $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

It DOES!!!

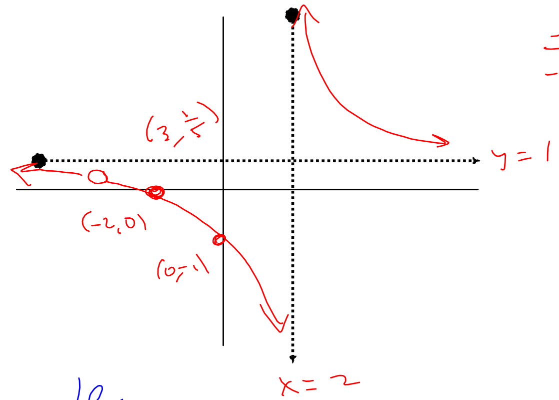
Removable Discontinuities:

They's a hole in the bucket, Deah Liza.

$$\frac{x^2+5x+6}{x^2+x-6} = \frac{(x+2)\cancel{(x+3)}}{\cancel{(x+3)}(x-2)} = \frac{x+2}{x-2} \quad (\text{if } x \neq -3)$$

$$\frac{-3+2}{-3-2} = \frac{-1}{-5} = \frac{1}{5}$$

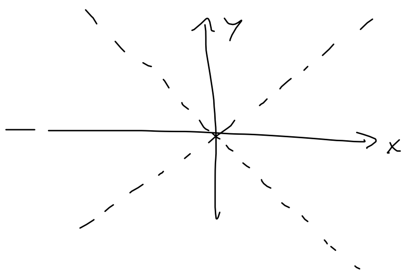
OLD



– Not Removable:

NEW

$$f(x,y) = \frac{xy}{x^2-y^2}$$



$x^2-y^2=0$

$x^2=y^2$

$\sqrt{x^2}=\sqrt{y^2}$

$|x|=|y|$

$x=\pm y$

$$\frac{x^4-y^4}{x^2+y^2} = \frac{\cancel{(x^2+y^2)}(x^2-y^2)}{\cancel{x^2+y^2}} = x^2-y^2$$

$(\text{if } (x,y) = (0,0))$

Final word

Vector notation collapses everything back to Calc I, symbolically

$\lim_{\bar{x}\to\bar{x}_0} \bar{F}(\bar{x}) = L$ means

$\forall\ \varepsilon>0\ \exists\ \delta>0 \ni \|\bar{x}-\bar{x}_0\|<\delta$

$\Rightarrow \|\bar{F}(\bar{x})-L\|<\varepsilon.$