

Stokes' Theorem.

We generalize Green's Theorem.

Recall from Section 16.4:

GREEN'S THEOREM Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C . If P and Q have continuous partial derivatives on an open region that contains D , then

$$\bar{F} = \langle P, Q \rangle dy dx \quad \bar{r}(t) = \langle x(t), y(t) \rangle \quad \int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

I keep trying to express this as a "curl" sort of thing, bringing you guys back to the vector notation and the standard cross product we see, over and over, in the integrand.

$$P = P(x, y) = P(x(t), y(t))$$

$$Q = Q(x, y) = Q(x(t), y(t))$$

$$\bar{F} = \langle P, Q, 0 \rangle$$

$$\bar{r} = \bar{r}(t) = \langle x(t), y(t), 0 \rangle \Rightarrow d\bar{r} = \langle x'(t), y'(t), 0 \rangle dt$$

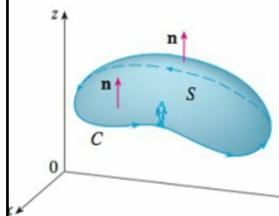
$$\text{curl } (\bar{F}) : \langle 0, 0, Q_x - P_y \rangle$$

$$\begin{aligned} & \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle, \frac{dx}{dt}, \frac{dy}{dt} \\ & \times \langle P, Q, 0 \rangle, P, Q \\ & \underline{\langle 0, 0, Q_x - P_y \rangle} \end{aligned}$$

$$\iint_D \bar{F} \cdot d\bar{S} = \iint_D \bar{F} \cdot \bar{n} dS$$

$$\frac{\bar{r}_u \times \bar{r}_v}{\|\bar{r}_u \times \bar{r}_v\|}$$

$$\int_C \bar{F} \cdot d\bar{r} = \iint_D \text{curl } \bar{F} \cdot d\bar{S}$$



STOKES' THEOREM Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let \mathbf{F} be a vector field whose components have continuous partial derivatives on an open region in \mathbb{R}^3 that contains S . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

$$\text{curl } \bar{F} =$$

FIGURE 1

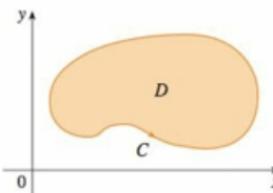
It's hard to keep my mouth shut (so I don't) about the integral on the left, because I KNOW we wanted to state GREEN'S THEOREM in this language. Now this curl stuff requires a 3-D vector field, \mathbf{F} . But we can make Green's work in 3-D just by adding a trivial 0 in the 3rd component of \mathbf{F} .

Stokes' Theorem INCLUDES Green's Theorem as a special case! So it's this huge sledgehammer that covers everything.

Since I don't know how to say this any better:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds \quad \text{and} \quad \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS$$

Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of \mathbf{F} is equal to the surface integral of the normal component of the curl of \mathbf{F} .



$$\iint_S \text{curl } \mathbf{F} \cdot (\bar{r}_u \times \bar{r}_v) dudv$$

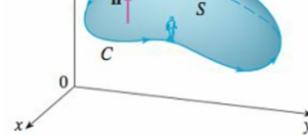
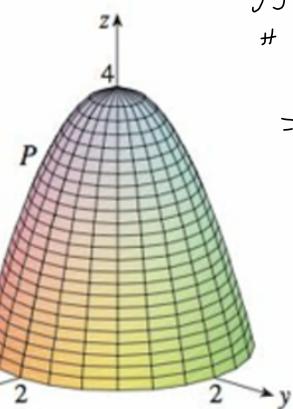
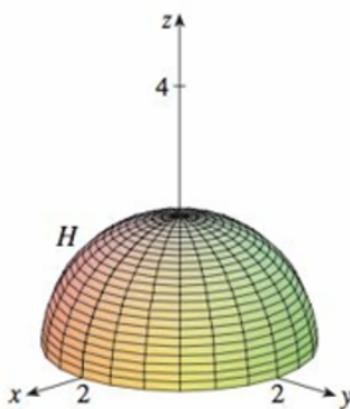


FIGURE 1

- I.** A hemisphere H and a portion P of a paraboloid are shown. Suppose \mathbf{F} is a vector field on \mathbb{R}^3 whose components have continuous partial derivatives. Explain why

$$\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$



$$\iint_H \operatorname{curl} \bar{\mathbf{F}} \cdot d\bar{\mathbf{S}} = \int_C \bar{\mathbf{F}} \cdot d\bar{r}$$

$$= \iint_P \operatorname{curl} \bar{\mathbf{F}} \cdot d\bar{\mathbf{S}}, \text{ by Stokes' Thm.}$$

2-6 Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

2. $\mathbf{F}(x, y, z) = 2y \cos z \mathbf{i} + e^x \sin z \mathbf{j} + xe^y \mathbf{k}$,
 S is the hemisphere $x^2 + y^2 + z^2 = 9, z \geq 0$, oriented upward

This exercise was worked in class.
For the video, check out the ZOOM recording for May 1st in the Course Shell.

$$\int_C \bar{\mathbf{F}} \cdot d\bar{r} = \int_C \bar{\mathbf{F}} \cdot \bar{T} d\bar{s} = \int_C \bar{\mathbf{F}} \cdot \bar{r}' dt$$

$$\begin{aligned} \bar{\mathbf{F}} &= \langle 2y \cos z, e^x \sin z, xe^y \rangle = \langle 2 \cdot 3 \sin t \cos 0, e^{3 \sin t} \sin 0, 3 \cos t e^{3 \sin t} \rangle \\ &= \langle 6 \sin t, 0, 3 \cos t e^{3 \sin t} \rangle. \end{aligned}$$

$$\bar{r}(t) = \langle 3 \cos t, 3 \sin t, 0 \rangle \quad 0 \leq t \leq 2\pi$$

$$\bar{r}'(t) = \langle -3 \sin t, 3 \cos t, 0 \rangle$$

$$\begin{aligned} \bar{\mathbf{F}} \cdot d\bar{r} &= \bar{\mathbf{F}}(\bar{r}(t)) \cdot \bar{r}'(t) dt \\ &= \langle 6 \sin t, 0, 3 \cos t e^{3 \sin t} \rangle \cdot \langle -3 \sin t, 3 \cos t, 0 \rangle dt \\ &= (-18 \sin^2 t + 0 + 0) dt = -18 \left(\frac{1 - \cos(2t)}{2} \right) dt \\ &= 9(\cos(2t) - 1) dt = (9 \cos(t) - 9) dt \end{aligned}$$

and $\int_0^{2\pi} (9 \cos t - 9) dt = \left[9 \sin t - 9t \right]_0^{2\pi} = -18\pi$

2-6 Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

3. $\mathbf{F}(x, y, z) = x^2 z^2 \mathbf{i} + y^2 z^2 \mathbf{j} + xyz \mathbf{k}$,

S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$, oriented upward

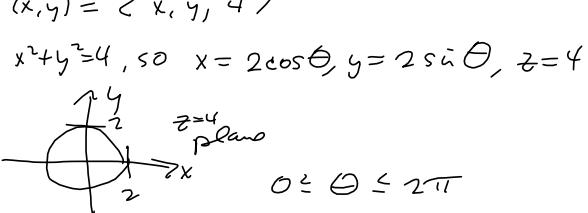
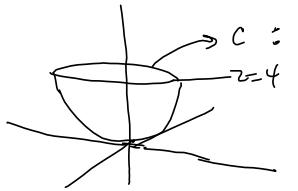
$$\iint_S \operatorname{curl} \bar{\mathbf{F}} \circ \bar{n} d\bar{S} = \iint_{S'} \operatorname{curl} \bar{\mathbf{F}} \circ \frac{\bar{r}_u \times \bar{r}_v}{\|\bar{r}_u \times \bar{r}_v\|} du dv$$

$$S' = \iint_{S'} \operatorname{curl} (\bar{\mathbf{F}}(r(u, v))) \circ \bar{r}_u \times \bar{r}_v du dv$$

But we don't need all that, thanks to Stokes!

He says:

$$\iint_S \operatorname{curl} \bar{\mathbf{F}} \circ d\bar{S} = \int_{C'} \bar{\mathbf{F}} \cdot d\bar{r} = \int_{C'} \bar{\mathbf{F}} \cdot (\bar{r}(t)) \bar{r}'(t) dt$$



$$\bar{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 4 \rangle$$

$$\bar{r}'(\theta) = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$$

$$\bar{F}(\bar{r}(\theta)) = \langle (2^2 \sin^2\theta)(4)^2, (2^2 \cos^2\theta)(4)^2, (2\cos\theta)(2\sin\theta)(4) \rangle$$

$$\bar{F}(\bar{r}(\theta)) = \langle 4^3 \sin^2\theta, 4^3 \cos^2\theta, 4^2 \sin\theta \cos\theta \rangle$$

$$\int_0^{2\pi} ((4^3 \sin^2\theta)(-2\sin\theta) + (4^3 \cos^2\theta)(2\cos\theta) + 0) d\theta$$

$$128 \int_0^{2\pi} (-\sin^3\theta + \cos^3\theta) d\theta =$$

scratch

$$\sin^3\theta = \sin\theta(1 - \cos^2\theta) = \sin\theta + \cos^2\theta(-\sin\theta)$$

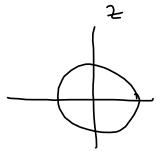
$$\cos^3\theta = \cos\theta(1 - \sin^2\theta) = \cos\theta - \sin^2\theta \cos\theta$$

$$= 128 \int_0^{2\pi} (-\sin\theta - \cos^2\theta(-\sin\theta) + \cos\theta - \sin^2\theta \cos\theta) d\theta$$

$$= 128 \left[\cos\theta - \frac{\cos^3\theta}{3} + \sin\theta - \frac{\sin^3\theta}{3} \right]_0^{2\pi}$$

$$= 128 \left[1 - \frac{1}{3} - \left(1 - \frac{1}{3} \right) + (0 - 0) - (0 - 0) \right] = 0 \quad \square$$

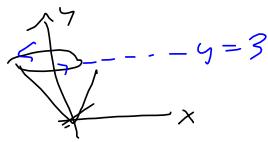
4. $\mathbf{F}(x, y, z) = x^2y^3z\mathbf{i} + \sin(xyz)\mathbf{j} + xyz\mathbf{k}$,
 S is the part of the cone $y^2 = x^2 + z^2$ that lies between the
planes $y = 0$ and $y = 3$, oriented in the direction of the
positive y -axis



$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$\mathbf{C} = d\mathbf{S}$$

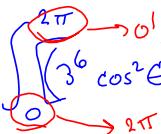
$$y = 3 \Rightarrow y^2 = 3^2 = x^2 + z^2 = 9$$



$$\mathbf{r}(\theta) = \langle 3 \cos \theta, 3, 3 \sin \theta \rangle$$

$$\mathbf{r}'(\theta) = \langle -3 \sin \theta, 0, 3 \cos \theta \rangle$$

$$\bar{\mathbf{F}}(\bar{\mathbf{r}}(\theta)) = \langle (3^2 \cos^2 \theta)(3^3)(3 \sin \theta), (3^3 \cos \theta \sin \theta), 3^3 \cos \theta \sin \theta \rangle$$



CLOCKWISE!

$$= \int_0^{2\pi} [-3^7 \cos^2 \theta \sin^2 \theta + 3^4 \cos^2 \theta \sin \theta] d\theta$$

$$\text{scratch: } \frac{1}{4} (1 + \cos 2\theta)(1 - \cos 2\theta)$$

$$= \frac{1}{4} [1 - \cos^2 2\theta] = \frac{1}{4} - \frac{1}{4} [\frac{1}{2} (1 + \cos 4\theta)]$$

$$= \frac{1}{4} - \frac{1}{8} - \frac{1}{8} \cos 4\theta$$

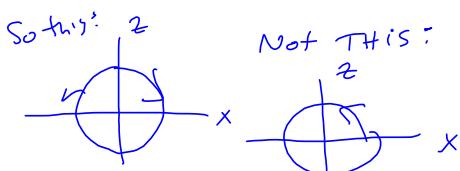
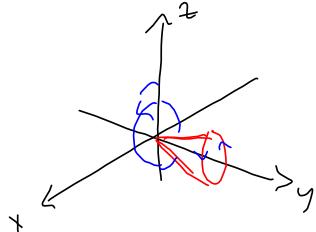
$$= -3^7 \cdot \frac{1}{8} [1 - \cos 4\theta]$$

$$= -\frac{3^7}{8} \int_0^{2\pi} (1 - \cos 4\theta) d\theta + 3^4 \left[\frac{\cos^3 \theta}{3} \right]_0^{2\pi}$$

$$= -\frac{3^7}{8} \left[\theta \right]_0^{2\pi} - \frac{3^7}{8} \cdot \frac{1}{4} \int_0^{2\pi} \cos(4\theta) \cdot 4d\theta + 0$$

$$= -\frac{3^7}{8} \cdot 2\pi - \left[\frac{3^7}{32} \sin(4\theta) \right]_0^{2\pi} = -\frac{3^7}{4} \pi = -\frac{2187}{4} \pi$$

This is the wrong sign. Look at orientation



Clockwise w/

$$x = 3 \cos \theta, z = 3 \sin \theta$$

$$\text{so } \int_{2\pi}^0 \text{ or}$$

$$\mathbf{r} = \langle 3 \sin \theta, 0, 3 \cos \theta \rangle$$

would also do it for us.

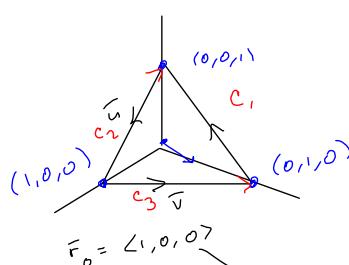
$\int_{C_1} \dots$

7-10 Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.

Done in class. See Lecture Recording.

$$\mathbf{F}(x, y, z) = (x + y^2) \mathbf{i} + (y + z^2) \mathbf{j} + (z + x^2) \mathbf{k},$$

C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$



$\int_C \bar{\mathbf{F}} \cdot d\bar{\mathbf{r}}$ Need to build 3 lines
to eval 3 line integrals, w/o
STOKES.

Eg'm of plane:

$$\bar{u} = \langle -1, 0, 1 \rangle, \bar{-1, 0}$$

$$\times \bar{v} = \langle -1, 1, 0 \rangle, \bar{-1, 1}$$

$$\langle -1, -1, -1 \rangle = \bar{n}$$

$$\bar{n} \cdot \langle x-1, y, z \rangle = 0$$

$$-1(x-1) - 1(y) - 1z = 0$$

$$-x + 1 - y - z = 0$$

$$z = 1-x-y \text{ or } x+y+z = 1$$

\bar{n} characterizes the surface

$$\text{Now, } \iint_S \bar{\mathbf{F}} \cdot d\bar{\mathbf{S}} =$$

$$\iint_S \bar{\mathbf{F}} \cdot \bar{n} dA =$$



$$\int_0^1 \int_0^{1-x} (\text{curl } \bar{\mathbf{F}}) \cdot \bar{r}_x \times \bar{r}_y dy dx$$

$$\bar{r} : x = y, z = 1-x-y$$

$$\bar{r}_x = \langle 1, 0, -1 \rangle, \bar{1, 0}$$

$$\times \bar{r}_y = \langle 0, 1, -1 \rangle, \bar{0, 1}$$

$$\langle 1, 1, 1 \rangle = \bar{r}_x \times \bar{r}_y$$

$$\text{scratch: curl } \bar{\mathbf{F}} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$$

$$\nabla \times \bar{\mathbf{F}} = \langle x+y^2, y+z^2, z+x^2 \rangle, x+y^2, y+z^2$$

$$\langle 2z, 2x, 2y \rangle = \text{curl } \bar{\mathbf{F}}$$

$$= \int_0^1 \int_0^{1-x} \langle 2z, 2x, 2y \rangle \cdot \langle 1, 1, 1 \rangle dy dx$$

$$= \int_0^1 \int_0^{1-x} \langle 2(1-x-y), 2x, 2y \rangle \cdot \langle 1, 1, 1 \rangle dy dx$$

$$= \int_0^1 \int_0^{1-x} (2-2x-2y+2x+2y) dy dx = \int_0^1 \int_0^{1-x} 2 dy dx = \int_0^1 [2y]_0^{1-x} dx$$

$$= 2 \int_0^1 (1-x) dx = 2 \left[x - \frac{1}{2}x^2 \right]_0^1 = 2 \left[1 - \frac{1}{2} \right] = 1 !$$

STOKES' THEOREM Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let \mathbf{F} be a vector field whose components have continuous partial derivatives on an open region in \mathbb{R}^3 that contains S . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \bar{\mathbf{F}} \cdot d\bar{\mathbf{S}}$$

$$d\bar{\mathbf{S}} = \bar{n} dS = \bar{r}_x \times \bar{r}_y dy dx$$

$$= \frac{\bar{r}_x \times \bar{r}_y}{||\bar{r}_x \times \bar{r}_y||} ||\bar{r}_x \times \bar{r}_y|| dy dx$$

13-15 Verify that Stokes' Theorem is true for the given vector field \mathbf{F} and surface S .

$$\textcircled{1} \quad \iint_S \operatorname{curl} \bar{\mathbf{F}} \cdot d\bar{S} = \textcircled{2} \int_C \bar{\mathbf{F}} \cdot d\bar{r}$$

13. $\mathbf{F}(x, y, z) = y^2 \mathbf{i} + x \mathbf{j} + z^2 \mathbf{k}$,

S is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$, oriented upward



$$\bar{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, r^2 \rangle$$

$$\bar{\mathbf{F}}(\bar{r}, \theta) = \langle r^2 \sin^2 \theta, r \cos \theta, r^4 \rangle$$

is premade

$$\begin{aligned} \textcircled{1} \quad & \nabla \times \bar{\mathbf{F}} \\ & \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \\ & \times \left\langle y^2, x, z^2 \right\rangle \times \left\langle y^2, x, z^2 \right\rangle \\ & \left\langle 0, 0, 1 - 1 \right\rangle = \frac{1}{0} \end{aligned}$$

$$\iint_S \operatorname{curl} \bar{\mathbf{F}} \cdot d\bar{S} = \iint_D \bar{\mathbf{F}} \cdot d\bar{S} = 0$$

$$\begin{aligned} \textcircled{2} \quad & \int_C \bar{\mathbf{F}} \cdot d\bar{r} \\ & \bar{r}(\theta) = \langle \cos \theta, \sin \theta, 1 \rangle \\ & \bar{r}'(\theta) = \langle -\sin \theta, \cos \theta, 0 \rangle \\ & \bar{\mathbf{F}}(\bar{r}(\theta)) = \langle \sin^2 \theta, \cos \theta, 1 \rangle \\ & \int_0^{2\pi} (\sin^2 \theta \cos \theta + \sin \theta \cos \theta) d\theta \\ & = \left[-\frac{\sin^3 \theta}{3} + \frac{\sin^2 \theta}{2} \right]_0^{2\pi} = 0 \end{aligned}$$

15. $\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$, $y = \pm\sqrt{1-(x^2+z^2)} = \pm\sqrt{1-r^2}$ take +
 S is the hemisphere $x^2 + y^2 + z^2 = 1$, $y \geq 0$, oriented in the direction of the positive y -axis

$$\iint_S \mathbf{curl} \bar{\mathbf{F}} \cdot d\mathbf{S} = \int_C \bar{\mathbf{F}} \cdot d\mathbf{r}$$

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$$

$$\langle x, y, z \rangle \cdot \langle y, z \rangle$$

$$\langle 0, -1, -1, -1 \rangle = \langle -1, -1, -1 \rangle = \mathbf{curl} \bar{\mathbf{F}}$$

$$\bar{r}(r, \theta) = \langle r \cos \theta, \sqrt{1-r^2}, r \sin \theta \rangle$$

from 2π down to zero

$$\bar{r}_r = \langle \cos \theta, \frac{1}{2}(1-r^2)^{\frac{1}{2}}(-2r), \sin \theta \rangle$$

$$= \langle \cos \theta, \frac{-r}{\sqrt{1-r^2}}, \sin \theta \rangle$$

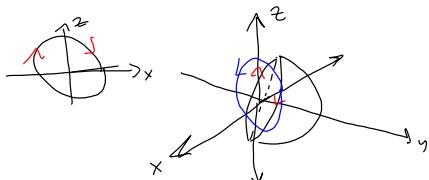
Better might be

$$\langle r \sin \theta, -\frac{r}{\sqrt{1-r^2}}, r \cos \theta \rangle$$

$0 \leq \theta \leq 2\pi$, measured from positive z -axis

① Due to orientation, the flag is going clockwise in the xz -plane, so 2π to 0

$$\int_0^0 \langle -1, -1, -1 \rangle \cdot \langle \cos \theta, \frac{-r}{\sqrt{1-r^2}}, \sin \theta \rangle dr$$



$$\bar{r}_r = \langle \cos \theta, \frac{-r}{\sqrt{1-r^2}}, \sin \theta \rangle, \cos \theta, \frac{-r}{\sqrt{1-r^2}}$$

$$y \bar{r}_\theta = \langle -r \sin \theta, 0, r \cos \theta \rangle, -r \cos \theta, 0$$

$$= \langle \frac{r^2 \cos \theta}{\sqrt{1-r^2}}, -r \sin^2 \theta - r \cos^2 \theta, \frac{-r^2 \sin \theta}{\sqrt{1-r^2}} \rangle$$

$$= \langle \frac{-r^2 \cos \theta}{\sqrt{1-r^2}}, -r, \frac{-r^2 \sin \theta}{\sqrt{1-r^2}} \rangle$$

$$\int_0^0 \int_0^{2\pi} \mathbf{curl} \bar{\mathbf{F}} \cdot \bar{r}_r \bar{r}_\theta dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 \langle -1, -1, -1 \rangle \cdot \langle \frac{-r^2 \cos \theta}{\sqrt{1-r^2}}, -r, \frac{-r^2 \sin \theta}{\sqrt{1-r^2}} \rangle dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(\frac{r^2 \cos \theta}{\sqrt{1-r^2}} + r + \frac{r^2 \sin \theta}{\sqrt{1-r^2}} \right) dr d\theta$$

$$= \int_0^{2\pi} (\cos \theta \sin \theta) d\theta \int_0^1 \frac{r^2}{\sqrt{1-r^2}} dr + \int_0^{2\pi} \int_0^1 r dr d\theta$$

$$= [\sin \theta - \cos \theta]_0^{2\pi} + \int_0^{2\pi} \left[\frac{1}{2} r^2 \right]_0^1 d\theta$$

$$= 0 + \int_0^{2\pi} \frac{1}{2} d\theta$$

$$= 0 + \frac{1}{2} [2\pi] = \pi$$

$$(2) \int_C \bar{\mathbf{F}} \cdot d\mathbf{r} =$$

$$\bar{r}(\theta) = \langle \cos \theta, 0, \sin \theta \rangle$$

$$\bar{F} = \langle y, z, x \rangle = \langle 0, \sin \theta, \cos \theta \rangle$$

$$\bar{r}'(\theta) = \langle -\sin \theta, 0, \cos \theta \rangle$$

$$\bar{F} \cdot \bar{r}'(\theta) = 0 + 0 + \cos^2 \theta$$

$$\bar{r}'(\theta) = \langle -\sin \theta, 0, \cos \theta \rangle$$

$$= \frac{1}{2} + \frac{1}{2} \cos(2\theta)$$

$$\bar{F}(\bar{r}(\theta)) \cdot \bar{r}'(\theta)$$

$$= \langle 0, \cos \theta, \sin \theta \rangle \cdot \langle \cos \theta, 0, -\sin \theta \rangle$$

$$= \langle 0, \cos \theta, \sin \theta \rangle \cdot \langle \cos \theta, 0, -\sin \theta \rangle$$

$$= - \int_0^{2\pi}$$

$$\int_{2\pi}^0 \left(\frac{1}{2} + \frac{1}{2} \cos(2\theta) \right) d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{1}{4} \sin(2\theta) \right]_{2\pi}^0$$

$$= -\frac{1}{2} \cdot 2\pi = -\pi$$

so I messed up the orientation somehow.

16. Let C be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z \, dx - 2x \, dy + 3y \, dz = \int_Q \bar{F} \cdot d\bar{r} = \iint_S \operatorname{curl} \bar{F} \cdot d\bar{S}$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

$$\bar{F} = \langle z, -2x, 3y \rangle$$

$$\bar{r} = \langle x, y, 1-x-y \rangle$$

$$\bar{r}_x = \langle 1, 0, -1 \rangle, \bar{r}_y = \langle 0, 1, 0 \rangle$$

Scratch:

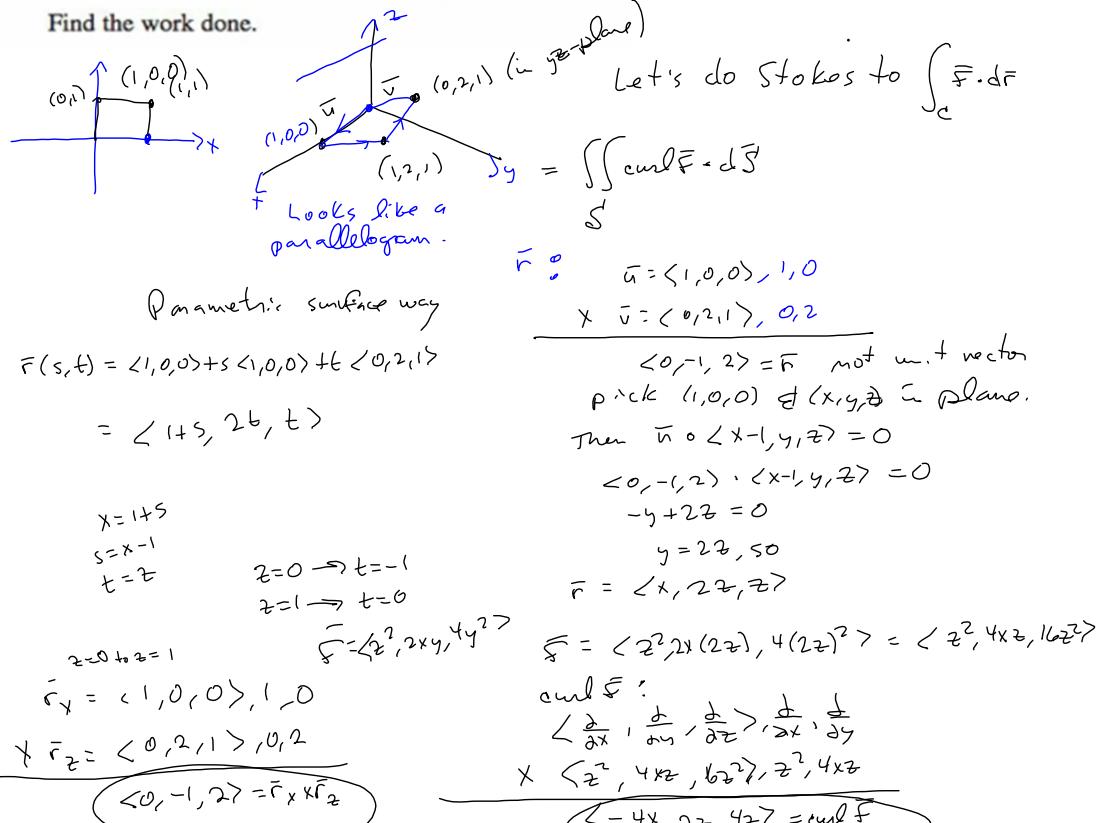
$$\begin{aligned} & \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \\ & \times \left\langle z, -2x, 3y \right\rangle, \bar{r}_x \cdot \bar{r}_y \\ & \underline{\left\langle 3, 1, -2 \right\rangle = \operatorname{curl} \bar{F}} \quad \underline{\left\langle 1, 1, 1 \right\rangle = \bar{r}_x \times \bar{r}_y} \end{aligned}$$

$$\iint_S \operatorname{curl} \bar{F} \cdot d\bar{S} = \int_{x=a}^{x=b} \int_{y=c}^{y=d} 2 \, dy \, dx = 2 \text{ Area of } S \quad \square$$

17. A particle moves along line segments from the origin to the points $(1, 0, 0)$, $(1, 2, 1)$, $(0, 2, 1)$, and back to the origin under the influence of the force field

$$\mathbf{F}(x, y, z) = z^2 \mathbf{i} + 2xy \mathbf{j} + 4y^2 \mathbf{k}$$

Find the work done.



If I can avoid evaluating
4 line integrals, that'd be
good!

Let's do Stokes to $\int_C \bar{F} \cdot d\bar{r}$

$$\iint_S \text{curl } \bar{F} \cdot d\bar{S}$$

$$\bar{r} \circ \bar{u} = \langle 1, 0, 0 \rangle, 1, 0$$

$$\times \bar{v} = \langle 0, 2, 1 \rangle, 0, 2$$

$\langle 0, -1, 2 \rangle = \bar{r}$ not unit vector
pick $\langle 1, 0, 0 \rangle \oplus \langle x, y, z \rangle \in \text{plane}$.

Then $\bar{u} \circ \langle x-1, y, z \rangle = 0$

$$\langle 0, -1, 2 \rangle \cdot \langle x-1, y, z \rangle = 0$$

$$-y+2z=0$$

$$y=2z, \text{ so}$$

$$\bar{r} = \langle x, 2z, z \rangle$$

$$\bar{F} = \langle z^2, 2x(2z), 4(2z)^2 \rangle = \langle z^2, 4xz, 16z^2 \rangle$$

$\text{curl } \bar{F} :$

$$\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

$$\times \langle z^2, 4xz, 16z^2 \rangle, z^2, 4xz$$

$$\langle -4x, 2z, 4z \rangle = \text{curl } \bar{F}$$

$$\text{curl } \bar{F} \circ \bar{r}_s \times \bar{r}_t = 0 - 2z + 8z = 6z$$

$$\iint_S \dots = \int_0^1 \int_0^1 6z \, dz \, dx = 6 \int_0^1 dx \int_0^1 z \, dz$$

$$= 6 \left[\frac{z^2}{2} \right]_0^1 \left[\frac{1}{2} z^2 \right]_0^1 = 6 \left[\frac{1}{2} \right] \left[\frac{1}{2} \right] = 6 \left[\frac{1}{2} \right] \left[\frac{1}{2} \right]$$

$$= 3$$

$$\bar{r}(s, t) = \langle 1+s, 2t, t \rangle$$

$$\bar{r}_s = \langle 1, 0, 0 \rangle, 1, 0$$

$$\times \bar{r}_t = \langle 0, 2, 1 \rangle, 0, 2$$

$$\langle 0, -1, 2 \rangle = \bar{r}_s \times \bar{r}_t$$

$$\langle -4x, 2z, 4z \rangle = \text{curl } \bar{F}$$

$$\Rightarrow \text{curl } \bar{F}(s, t) = \langle -4-4s, 2t, 4t \rangle$$

$$\text{curl } \bar{F} \circ \bar{r}_s \times \bar{r}_t = 0 - 2t + 8t = 6t$$

$$\int \int (6t) \, ds \, dt$$

$$= \int_{-1}^0 \int_0^1 6t \, ds \, dt$$

$$= \left[\int_{-1}^0 6t \, ds \right] \left[\int_0^1 dt \right]$$

$$= \left(3t^2 \Big|_0^1 \right) \left(t \Big|_{-1}^0 \right)$$

$$= (3-0)(0-(-1)) = 3$$

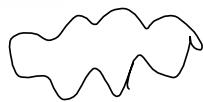
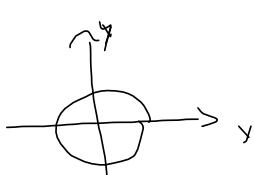
18. Evaluate

$$\vec{F} = \langle y + \sin x, z^2 + \cos y, x^3 \rangle$$

$$\int_C (y + \sin x) dx + (z^2 + \cos y) dy + x^3 dz$$

where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle, 0 \leq t \leq 2\pi$.[Hint: Observe that C lies on the surface $z = 2xy$.]

$$z = 2xy$$



$$\vec{r}(t) = \langle r \sin t, r \cos t, 2r^2 \sin t \cos t \rangle$$

$$\vec{r}_r = \langle \sin t, \cos t, 2r \sin t \cos t \rangle$$

$$\begin{aligned}\vec{r}_t &= \langle r \cos t, -r \sin t, 2r^2 (\cos^2 t - \sin^2 t) \rangle \\ &= \langle r \cos t, -r \sin t, 2r^2 - 4r^2 \sin^2 t \rangle\end{aligned}$$

$$1 - \sin^2 t - \sin^2 t$$

$$= 1 - 2 \sin^2 t$$



$$\vec{r}_r = \langle \sin t, \cos t, 2r \sin t \cos t \rangle, \sin t, \cos t$$

$$\vec{r}_t = \langle r \cos t, -r \sin t, 2r^2(1 - 2 \sin^2 t) \rangle, r \cos t, -r \sin t$$

$$\langle 2r^2 \cos t(1 - 2 \sin^2 t) + 2r^2 \sin^2 t \cos t, 2r^2 \cos^2 t \sin t, -r \sin^2 t - r \cos^2 t \rangle$$

$$= \langle 2r^2 \cos t - 2r^2 \sin^2 t, 2r^2 \cos^2 t \sin t, -r \rangle = \vec{r}_r \times \vec{r}_t$$

$$\vec{F} = \langle y + \sin x, z^2 + \cos y, x^3 \rangle$$

$$\text{curl } \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$$

$$\begin{array}{c} \text{red blob} \\ \times \quad y + \sin x, z^2 + \cos y, x^3, \quad \text{red X} \end{array}$$

$$\langle 0, -2z, 3x^2, 1 \rangle = \text{curl } \vec{F}$$

$$\text{curl } \vec{F} = \boxed{\langle -2(2r^2 \sin t \cos t), 3r^2 \sin t, 1 \rangle}$$

$$\boxed{\langle 2r^2 \cos t - 2r^2 \sin^2 t, 2r^2 \cos^2 t \sin t, -r \rangle} = \vec{r}_r \times \vec{r}_t$$

=

$$\int_0^{2\pi} \int_0^1$$

- 19.** If S is a sphere and \mathbf{F} satisfies the hypotheses of Stokes' Theorem, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

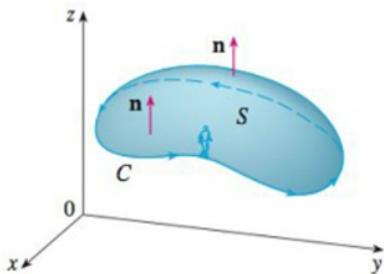
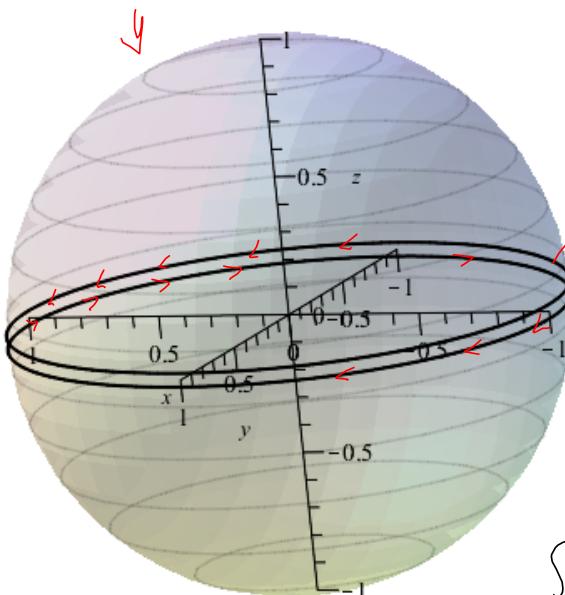


FIGURE 1

STOKES' THEOREM Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let \mathbf{F} be a vector field whose components have continuous partial derivatives on an open region in \mathbb{R}^3 that contains S . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$



S' = sphere outward

S_1 = Top half upward

S_2 = Bottom half, downward

Looking down, we traverse C_1 , counter-clockwise

traverse C_2 , looking up
make it counter-clockwise,

$$\iint_S \operatorname{curl} \bar{\mathbf{F}} \cdot d\bar{\mathbf{S}} = \sum_{k=1}^2 \iint_{S_k} \operatorname{curl} \bar{\mathbf{F}} \cdot d\bar{\mathbf{S}}$$

$$= \iint_{S_1} \operatorname{curl} \bar{\mathbf{F}} \cdot d\bar{\mathbf{S}} + \iint_{S_2} \operatorname{curl} \bar{\mathbf{F}} \cdot d\bar{\mathbf{S}} = \int_S \bar{\mathbf{F}} \cdot d\bar{\mathbf{r}} + \int_{C_2} \bar{\mathbf{F}} \cdot d\bar{\mathbf{r}}$$

$$C_2 = -C_1 \Rightarrow \text{the previous is } \int_{C_1} \bar{\mathbf{F}} \cdot d\bar{\mathbf{r}} + \int_{-C_1} \bar{\mathbf{F}} \cdot d\bar{\mathbf{r}}$$

$$= \int_{C_1} \bar{\mathbf{F}} \cdot d\bar{\mathbf{r}} - \int_{C_1} \bar{\mathbf{F}} \cdot d\bar{\mathbf{r}} = 0$$

20. Suppose S and C satisfy the hypotheses of Stokes' Theorem and f, g have continuous second-order partial derivatives. Use Exercises 24 and 26 in Section 16.5 to show the following.

$$(a) \int_C (f \nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$$

$$(b) \int_C (f \nabla f) \cdot d\mathbf{r} = 0$$

$$(c) \int_C (f \nabla g + g \nabla f) \cdot d\mathbf{r} = 0$$

Not too interested in your doing this. It's all good, but we're on a time line, here.

$$\int_C f \nabla g \cdot d\mathbf{r} = \iint_S \text{curl}(f \nabla g) \cdot d\mathbf{S}'$$

$$f = (x, y, z), \quad g = (u, v, w)$$

$$f \nabla g = \langle f u_x, f v_y, f w_z \rangle$$

$$\text{curl}(f \nabla g) = \nabla \times f \nabla g$$

$$\begin{aligned} & \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \\ &= x \left\langle f u_x, f v_y, f w_z \right\rangle \left\langle f u_x, f v_y \right\rangle \\ & \quad \overline{\left\langle \frac{\partial}{\partial y}(f w_z) - \frac{\partial}{\partial z}(f v_y), \frac{\partial}{\partial z}(f u_x) - \frac{\partial}{\partial x}(f w_z), \frac{\partial}{\partial x}(f v_y) - \frac{\partial}{\partial y}(f u_x) \right\rangle} \end{aligned}$$

$$= f_y w_z + f w_{zy} - f_z v_y - f v_{yz}, f_z u_x + f u_{xz} - f_x w_z - f w_{zx}, f_x v_y + f v_{yx} - f_y u_x - f u_{xy} \rangle$$

