49. Let **F** be an inverse square field, that is, $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3$ for some constant c, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Show that the flux of **F** across a sphere S with center the origin is independent of the radius of S.

$$\iint_S \bar{F}\cdot d\bar{S} = \iint_S \bar{F}\cdot\bar{n}\, dS = \iint_S \bar{F}\cdot(\bar{r}_\phi\times\bar{r}_\theta)\, dA$$

$$\bar{F}(\bar{r}) = \frac{c\,\bar{r}}{\|\bar{r}\|^3}$$

Spherical Coordinates

$$\bar{r} = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle$$

$$\bar{r}_\phi = \langle \rho\cos\phi\cos\theta, \rho\cos\phi\sin\theta, \cancel{-\rho\sin\phi}\rangle$$

$$\bar{r}_\theta = \langle -\rho\sin\phi\sin\theta, \rho\sin\phi\cos\theta, \cancel{0}\rangle$$

$$\langle \rho^2\sin^2\phi\cos\theta, -(0-\rho^2\sin^2\phi\sin\theta), \rho^2\cos\phi\sin\phi\cos^2\theta+\rho^2\cos\phi\sin\phi\sin^2\theta\rangle$$

$$= \langle \rho^2\sin^2\phi\cos\theta, \rho^2\sin^2\phi\sin\theta, \rho^2\cos\phi\sin\phi\rangle = \bar{r}_\phi\times\bar{r}_\theta$$

$$= \iint_S \bar{F}\cdot(\bar{r}_\phi\times\bar{r}_\theta)\, dA = \iint_S \frac{c\,\bar{r}}{\|\bar{r}\|^3}\cdot\langle \rho^2\sin^2\phi\cos\theta, \rho^2\sin^2\phi\sin\theta, \rho^2\cos\phi\sin\phi\rangle$$

$$= \frac{c}{\rho^3}\iint_S \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle\cdot\langle \rho^2\sin^2\phi\cos\theta, \rho^2\sin^2\phi\sin\theta, \rho^2\cos\phi\sin\phi\rangle\, dA$$

$$= \frac{c}{\rho^3}\iint_S \left(\rho^3\sin^3\phi\cos^2\theta + \rho^3\sin^3\phi\sin^2\theta + \rho^3\cos^2\phi\sin\phi\right) dA$$

$$= c\iint_S \left(\sin^3\phi + \cos^2\phi\sin\phi\right) dA = c\iint_S \sin\phi\left(\sin^2\phi + \cos^2\phi\right) dA$$

$$= c\iint_S \sin\phi\, dA = c\int_0^\pi\int_0^{2\pi} \sin\phi\, d\theta\, d\phi = c\int_0^\pi \sin\phi\,\left[\theta\right]_0^{2\pi} d\phi$$

$$= 2\pi c\int_0^\pi \sin\phi\, d\phi = -2\pi c\left[\cos\phi\right]_0^\pi$$

$$= -2\pi c\left[\cos\pi - \cos 0\right] = -2\pi c\left[-1-1\right] = \boxed{4\pi c}$$

Nothing involving ρ, here