CURL AND DIVERGENCE

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k} = \langle P, Q, R \rangle$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} = \nabla \times \mathbf{F}$$

This should make you think of torque, and curl does, indeed, say something about the tendency of things to rotate.

3 THEOREM If f is a function of three variables that has continuous second-order partial derivatives, then

$$\operatorname{curl}(\nabla f) = \mathbf{0}$$

Since a conservative vector field is one for which $\mathbf{F} = \nabla f$, Theorem 3 can be rephrased as follows:

If $\mathbf{F}$ is conservative, then $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

This gives us a way of verifying that a vector field is not conservative.

The converse of Theorem 3 is not true in general, but the following theorem says the converse is true if $\mathbf{F}$ is defined everywhere.

4 THEOREM If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl} \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

That's why we spend a lot of time looking for holes. But just because curl(**F**) is zero doesn't automatically mean that **F** is conservative. But if it's defined everywhere, with continuous second partials everywhere, then yes.

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This is the first example where a conservative field **F** was a function of 3 variables. Part (b) takes it to the next (3-variable) level.

EXAMPLE 3

(a) Show that

$$\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$$

is a conservative vector field.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

(a) $\bar{F}$ has cnt's partials everywhere $\Rightarrow$ T4 applies, i.e.,

$\mathrm{Curl}(\bar{F}) = \bar{0} \Rightarrow$ Conservative

$\nabla \times \bar{F} = \mathrm{Curl}\,\bar{F}$ $\quad \langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \rangle\ \frac{d}{dx},\ \frac{d}{dy}$

$\times\ \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle\ y^2z^3,\ 2xyz^3$

$\langle 6xyz^2 - 6xyz^2,\ 3y^2z^2 - 3y^2z^2,\ 2yz^3 - 2yz^3 \rangle$

$= \langle 0, 0, 0 \rangle = \bar{0}$ ✓

(b) $f_x = y^2z^3$

$\Rightarrow f = \int y^2z^3\,dx + g(y,z) = xy^2z^3 + g(y,z)$

$\Rightarrow f_y = 2xyz^3 + \frac{d}{dy}(g(y,z)) = 2xyz^3$

$\Rightarrow \frac{d}{dy}(g(y,z)) = 0 \Rightarrow g(y,z) = g(z)$

Take Stock: $f = xy^2z^3 + g(z)$

$f_z = 3xy^2z^2 + g'(z) = 3xy^2z^2$

$\Rightarrow g'(z) = 0 \Rightarrow g(z) = C \overset{\text{SET}}{=} 0$

$\therefore\ \boxed{f(x,y,z) = xy^2z^3}$

divergence of F

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \mathbf{F}$$

curl **F** is a vector field but div **F** is a scalar field.

11 THEOREM If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\operatorname{div} \operatorname{curl} \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0$$

(Apply Clairaut's 3 times.)

If $\mathbf{F}(x, y, z)$ is the velocity of a fluid (or gas), then div $\mathbf{F}(x, y, z)$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume. In other words, div $\mathbf{F}(x, y, z)$ measures the tendency of the fluid to diverge from the point (x, y, z). If div $\mathbf{F} = 0$, then $\mathbf{F}$ is said to be **incompressible**.

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$$

Laplace's equation

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

VECTOR FORMS OF GREEN'S THEOREM

$$\mathbf{r} = \overline{r} = \langle x(t), y(t) \rangle$$

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} = \langle P(x,y), Q(x,y)\rangle = \langle P(x(t),y(t)), Q(x(t),y(t))\rangle = \mathbf{F}(\mathbf{r})$$

F's line integral:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \oint_C \langle P(x(t),y(t)), Q(x(t),y(t))\rangle \bullet \langle x'(t), y'(t)\rangle dt$$

By the expedient of treating **F** as a vector field in 3-space, with 3rd entry 0, we can re-state Green's Theorem in vector notation as follows:

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} =$$

$$\nabla \times \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P(x,y), Q(x,y), 0 \rangle$$

$$= \left\langle \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(Q(x,y)), -\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(P(x,y))\right), \frac{\partial}{\partial x}(Q(x,y)) - \frac{\partial}{\partial y}(P(x,y)) \right\rangle$$

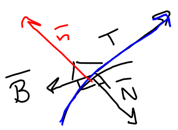
$$= \left\langle 0 - \frac{\partial Q}{\partial z}, -\left(0 - \frac{\partial P}{\partial z}\right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle = \langle 0, 0, Q_x - P_y \rangle$$

The line integral of the tangential component of **F** along C is the double integral of the vertical component of curl **F** over the region D enclosed by C.

That's a double mouthful. It's easier to express, symbolically:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl}\,\mathbf{F}) \cdot \mathbf{k}\, dA$$

$\overline{T} \times \overline{N}$



Recall the outward normal vector **n**.

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{j}$$

(Obtained by the easiest way to build a vector whose dot product with the unit tangent is 0.)

The inward normal is $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$, where $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

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$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b (\mathbf{F}\cdot\mathbf{n})(t)\,|\mathbf{r}'(t)|\,dt$$

$$= \int_a^b \left[\frac{P(x(t),y(t))\,y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t),y(t))\,x'(t)}{|\mathbf{r}'(t)|}\right]|\mathbf{r}'(t)|\,dt$$

$$= \int_a^b P(x(t),y(t))\,y'(t)\,dt - Q(x(t),y(t))\,x'(t)\,dt$$

$$= \int_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \iint_D \nabla\cdot\bar{F}\,dA$$

$$= \iint_D \operatorname{div}(\bar{F})\,dA$$

This last by Green's Thm.

This gives us a second vector form for Green's Thm:

6ed'n 13
See #35
(e #33)

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}(x,y)\,dA$$

outward

This version says that the line integral of the normal component of **F** along C is equal to the double integral of the divergence of **F** over the region D enclosed by C.

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1–8 Find (a) the curl and (b) the divergence of the vector field.

1. $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} - x^2y\,\mathbf{k}$

3. $\mathbf{F}(x, y, z) = \mathbf{i} + (x + yz)\,\mathbf{j} + (xy - \sqrt{z})\,\mathbf{k}$

7. $\mathbf{F}(x, y, z) = \langle \ln x, \ln(xy), \ln(xyz) \rangle$

1

(a) $\bar{F} = \langle xyz, 0, -x^2y \rangle$

$\text{curl } \bar{F} = \nabla \times \bar{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \; \frac{\partial}{\partial x} \; \frac{\partial}{\partial y}$

$\times \quad \langle xyz, 0, -x^2y \rangle \; xyz, 0$

$\langle -x^2 - 0, \quad xy - (-2xy), \quad 0 - xz \rangle = \boxed{\langle -x^2, 3xy, -xz \rangle}$

(b) $\nabla \cdot \bar{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle xyz, 0, -x^2y \rangle = \boxed{yz}$

13–18 Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

13. $\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$

$$\nabla \times F = \quad \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$$
$$\times \quad \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle\, y^2z^3, 2xyz^3$$
$$\langle 6xyz^2 - 6xyz^2, 3y^2z^2 - 3y^2z^2, 2yz^3 - 2yz^3 \rangle = 0$$

So ~~d~~ yes, by Thm 4

4 THEOREM If **F** is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then **F** is a conservative vector field.

$$f_x = y^2z^3 \Rightarrow f = \int y^2z^3\,dx + g(y,z) = xy^2z^3 + g(y,z)$$
$$f_y = 2xyz^3 = 2xyz^3 + g_y(y,z) \Rightarrow g_y(y,z) = 0 \Rightarrow$$
$$f_z = 3xy^2z^2 = 3xy^2z^2 + g'(z) \qquad g(y,z) = g(z), \text{actually}$$
$$\Rightarrow g'(z) = 0 \Rightarrow g(z) = \text{Constant} \stackrel{\text{SET}}{=} 0.$$

$$\boxed{f(x,y,z) = xy^2z^3}$$

17. $\mathbf{F}(x, y, z) = ye^{-x}\,\mathbf{i} + e^{-x}\,\mathbf{j} + 2z\,\mathbf{k}$

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$$
$$\times \langle ye^{-x}, e^{-x}, 2z \rangle\, ye^{-x}, e^{-x}$$
$$\langle 0, 0, -e^{-x} - e^{-x} \rangle \neq \vec{0}$$

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19. Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle x \sin y, \cos y, z - xy \rangle$? Explain.

No. If so, then $\operatorname{div}(\operatorname{Curl}(\bar{G})) = 0$

But $\operatorname{div}(\operatorname{curl}(\bar{G})) = \operatorname{div}(\langle x \sin y, \cos y, z - xy \rangle)$

$= \sin(y) - \sin(y) + 1 = 1 \neq 0$

which is impossible if G is a vector field

21. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$$

where f, g, h are differentiable functions, is irrotational

$$\operatorname{curl}(\bar{F}) = \left\langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right\rangle \frac{d}{dx}, \frac{d}{dy}$$

$$\times \langle f(x), g(y), h(z) \rangle f(x), g(y)$$

$\langle 0, 0, 0 \rangle \Rightarrow \bar{F}$ is irrotational.

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22. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(y, z)\,\mathbf{i} + g(x, z)\,\mathbf{j} + h(x, y)\,\mathbf{k}$$

is incompressible.

$\text{div}(\bar{F}) = \nabla \cdot \bar{F} \overset{?}{=} 0 = 0\bar{i} + 0\bar{j} + 0\bar{k}$ Yup.

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23–29 Prove the identity, assuming that the appropriate partial derivatives exist and are continuous. If f is a scalar field and $\mathbf{F}$, $\mathbf{G}$ are vector fields, then $f\mathbf{F}$, $\mathbf{F} \cdot \mathbf{G}$, and $\mathbf{F} \times \mathbf{G}$ are defined by

$$(f\mathbf{F})(x, y, z) = f(x, y, z)\, \mathbf{F}(x, y, z)$$
$$(\mathbf{F} \cdot \mathbf{G})(x, y, z) = \mathbf{F}(x, y, z) \cdot \mathbf{G}(x, y, z)$$
$$(\mathbf{F} \times \mathbf{G})(x, y, z) = \mathbf{F}(x, y, z) \times \mathbf{G}(x, y, z)$$

25. $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \nabla f$

$= \nabla \cdot \langle fP, fQ, fR \rangle$,

where $\bar{F} = \langle P, Q, R \rangle$

$= \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR)$

$= f_x P + \underline{f P_x} + f_y Q + \underline{\underline{f Q_y}} + f_z R + \underline{f R_z}$

$= f P_x + f Q_y + f R_z + f_x P + f_y Q + f_z R$

$= f\, \nabla \cdot \bar{F} + \bar{F} \cdot \nabla f$

$= f \operatorname{div}(\bar{F}) + \bar{F} \cdot \nabla f$

$\nabla f = \langle f_x, f_y, f_z \rangle$

"Scalar" "multiplication."

$= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle f(x, y, z)$

where "multiplication" is the differentiation operation suggested by their juxtaposition.

33. Use Green's Theorem in the form of Equation 13 to prove **Green's first identity**:

$$\iint_D f\nabla^2 g\, dA = \oint_C f(\nabla g)\cdot \mathbf{n}\, ds - \iint_D \nabla f \cdot \nabla g\, dA$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous. (The quantity $\nabla g \cdot \mathbf{n} = D_{\mathbf{n}}\, g$ occurs in the line integral. This is the directional derivative in the direction of the normal vector $\mathbf{n}$ and is called the **normal derivative** of g.)

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

By #25 $\operatorname{div}(f\vec F)$

$= \nabla f \cdot \vec F + f\,\nabla\cdot\vec F$

$= \nabla f\cdot\vec F + f\operatorname{div}(\vec F)$

$(fg)' = f'g + fg'$

$$\int_C f\,\nabla g\cdot\vec n\, ds = \iint_D \operatorname{div}(f\nabla g)\, dA = \iint_D (\nabla f\cdot\nabla g + f\operatorname{div}(\nabla g))\, dA$$

$$= \iint_D \nabla f\cdot\nabla g\, dA + \iint_D f\,\nabla\cdot\nabla g\, dA$$

$$= \iint_D \nabla f\cdot\nabla g\, dA + \iint_D f\,\nabla^2 g\, dA$$

$\Rightarrow$

$$\iint_D f\nabla^2 g\, dA = \int_C f\nabla g\cdot\vec n\, ds \boxed{- \iint_D \nabla f\cdot\nabla g\, dA}$$

35. Recall from Section 15.3 that a function g is called *harmonic* on D if it satisfies Laplace's equation, that is, $\nabla^2 g = 0$ on D. $= \nabla\cdot\nabla g$ Use Green's first identity (with the same hypotheses as in Exercise 33) to show that if g is harmonic on D, then $\oint_C D_{\mathbf{n}}\, g\, ds = 0$. Here $D_{\mathbf{n}}\, g$ is the normal derivative of g defined in Exercise 33.

$$\int_C \vec F\cdot d\vec r = \iint_D \operatorname{curl}(\vec F)\cdot\vec k\, dA$$

GREEN'S THEOREM Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$$

$$\int_C D_n g\, ds = \int_C 1\,\nabla g\cdot\vec n\, ds =$$

Let $f \equiv 1$ in #33, Then $\nabla f = 0$ and

Then $\displaystyle\int_C \nabla g\cdot n\, ds = \iint_D 1\,\nabla^2 g\, dA = 0$

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