$\boxed{6}$ **Theorem** Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

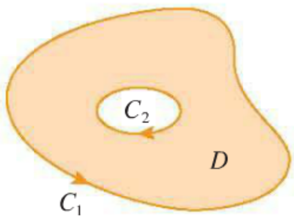

FIGURE 9

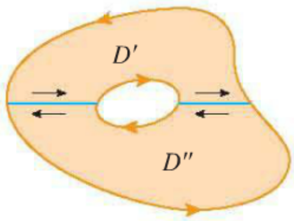

FIGURE 10

$$\vec{F} = \frac{1}{x^2+y^2}\langle -y, x\rangle$$

$$Q_x = -(x^2+y^2)^{-2}\,2x\cdot x + (x^2+y^2)^{-1}$$

$$= \frac{-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{x^2-2x^2+y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$P_y = -(x^2+y^2)^{-2}(2y)\cdot(-y) + (x^2+y^2)^{-1}(-1)$$

$$= \frac{2y^2-(x^2+y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = Q_x = P_y$$

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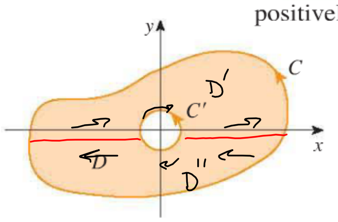
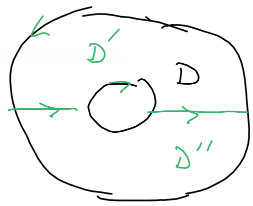


FIGURE 11

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$



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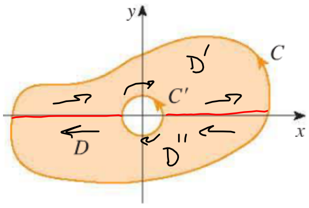

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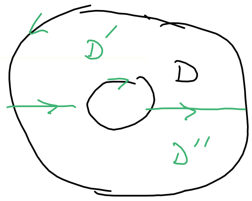
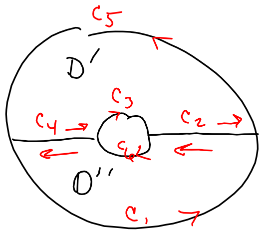


$Q_x = P_y$, here, with

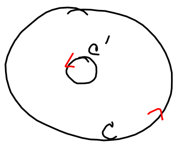
$P = \frac{-y}{x^2+y^2}$, $Q = \frac{x}{x^2+y^2}$



$\iint_{D'} = \iint_{D''} (Q_x - P_y)\,dA = 0$, b/c $Q_x = P_y$

Green's says

$$\iint_{D'} + \iint_{D''} = \cancel{\int_{c_2}} + \int_{c_5} + \cancel{\int_{c_4}} + \int_{c_3} + \int_{c_1} + \cancel{\int_{-c_2}} + \int_{c_6} + \cancel{\int_{-c_4}}$$



$$= \int_{c_5} + \int_{c_1} + C_{c_3} + C_6$$

$$= \int_c + \int_{-c'} = 0 \Rightarrow \int_c = -\int_{-c'} = \int_{c'}$$