

THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

Recall, FTC II: $\int_a^b F'(x) dx = F(b) - F(a)$

2 THEOREM Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C . Then



$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$\nabla f = \langle x_t, y_t, z_t \rangle$$

Basically, line integrals of conservative vector fields are independent of path.

And if it's independent of path, and the path is closed (ends where it started), then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

3 THEOREM $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D .

4 THEOREM Suppose \mathbf{F} is a vector field that is continuous on an open connected region D . If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D , then \mathbf{F} is a conservative vector field on D ; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

So, \mathbf{F} is conservative if and only if the corresponding line integrals are independent of path.

You want to show it's independent of path? Show it's conservative.

You want to show it's conservative? Prove it's independent of path (not always easy).

5 THEOREM If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D , then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

This is just Clairaut's Theorem.

If \mathbf{F} is a gradient for some f , then

this equation always holds.

$$\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$$

$$\mathbf{F} \text{ conservative} \Rightarrow \mathbf{F} = \nabla f$$

$$\mathbf{F} = \left\langle \frac{df}{dx}, \frac{df}{dy} \right\rangle = \langle P, Q \rangle$$

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

A simple curve is a curve that doesn't intersect itself anywhere between its endpoints.

A simply-connected region in the plane is a connected region D such

that every simple closed curve in D encloses only points that are in D . \Leftrightarrow No holes in D .

This is a fancy mathematical way of saying that D has no holes!

6 THEOREM Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D . Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

Then \mathbf{F} is conservative.

For MOST intents and purposes, Theorem 6 gives us a working converse to Theorem 5. You just want to be careful not to think everything's hunky-dory, when the domain has a hole, or the field is undefined somewhere inside the domain. All this theory requires the field to be nice and continuous (which implies bounded).

smooth

CONSERVATION OF ENERGY

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

So the work done by the force on the object is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t) dt$$

$$= \frac{m}{2} \int_a^b \frac{d}{dt} [\mathbf{r}'(t) \cdot \mathbf{r}'(t)] dt$$

(Theorem 14.2.3, Formula 4)

$$= \frac{m}{2} \int_a^b \frac{d}{dt} \|\mathbf{r}'(t)\|^2 dt = \frac{m}{2} [\|\mathbf{r}'(t)\|^2]_a^b$$

(Fundamental Theorem of Calculus)

$$= \frac{m}{2} (\|\mathbf{r}'(b)\|^2 - \|\mathbf{r}'(a)\|^2)$$

$$W = \frac{1}{2}m \|\mathbf{v}(b)\|^2 - \frac{1}{2}m \|\mathbf{v}(a)\|^2$$

$$K(t) = \frac{1}{2}m \|\mathbf{v}(t)\|^2 = \text{kinetic energy of the object.}$$

$$\frac{1}{2}mv^2 = \text{Kinetic Energy!}$$

$$W = K(B) - K(A) \quad \text{Work} = \text{the net change in kinetic energy.}$$

$$\text{potential energy} = P(x, y, z) = -f(x, y, z)$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla P \cdot d\mathbf{r} = -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))] = P(A) - P(B)$$

$$\vec{F} = \nabla f \quad \int_C \nabla f \cdot d\vec{r}$$

$$F = ma$$

Product Rule for Dot Product is just like ordinary

product

$$[fg]' = f'g + fg'$$

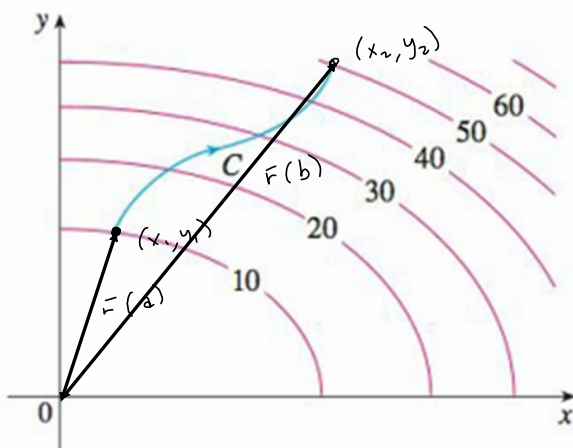
$$[\vec{r}' \cdot \vec{r}']' = \vec{r}'' \cdot \vec{r}' + \vec{r}' \cdot \vec{r}''$$

$$= \vec{r}'' \cdot \vec{r}' + \vec{r}' \cdot \vec{r}''$$

$$= 2 \vec{r}'' \cdot \vec{r}'$$

$$\Rightarrow \vec{r}'' \cdot \vec{r}' = \frac{1}{2} [\vec{r}' \cdot \vec{r}']'$$

1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.



$$\begin{aligned} \int_C \nabla f \cdot d\mathbf{r} &= f(F(b)) - f(F(a)) \\ &= \int_a^b \nabla f(F(t)) \cdot F'(t) dt \\ &= 50 - 10 = 40! \end{aligned}$$

3-10 Determine whether or not \mathbf{F} is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

7. $\mathbf{F}(x, y) = (ye^x + \sin y) \mathbf{i} + (e^x + x \cos y) \mathbf{j}$

By Thm 6:

$$P_y = e^x + \cos y$$

$$Q_x = e^x + \cos y$$

$\Rightarrow \mathbf{F}$ is conservative.

Variation of Parameters

$$f = \int f_x dx + g(y) = ye^x + x \sin y + g(y)$$

$$f_y = e^x + x \cos y + g'(y) = Q = e^x + x \cos y \Rightarrow$$

$$g'(y) = 0$$

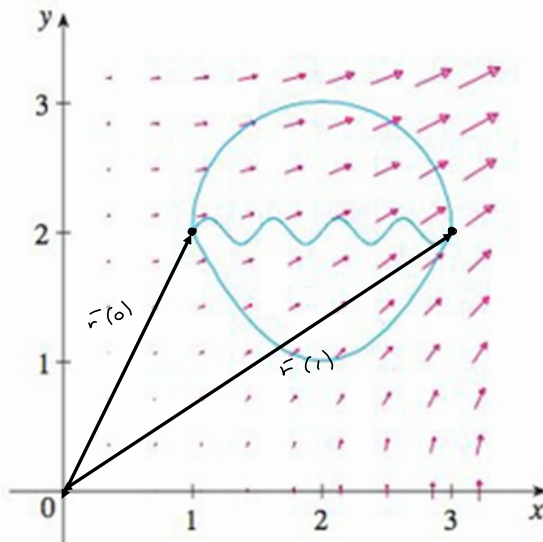
$$\Rightarrow g(y) = C$$

$$f(x, y) = ye^x + x \sin y + C \quad \forall C \in \mathbb{R}$$

II. The figure shows the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2 \rangle$ and three curves that start at $(1, 2)$ and end at $(3, 2)$.

(a) Explain why $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for all three curves.

(b) What is this common value?



$$P_y = 2x = Q_x \Rightarrow \text{conservative.}$$

$$f_x = 2xy$$

$$\Rightarrow f = x^2y + g(y)$$

$$f_y = x^2 \Rightarrow$$

$$f = x^2y + g(x)$$

$$g(x) = g(y) \Rightarrow g(x) = C$$

$$\text{Choose } C = 0 = g(y)$$

$$\text{Then } \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= f(\mathbf{r}(1)) - f(\mathbf{r}(0))$$

guessimate from the field

$$= f(3, 2) - f(1, 2)$$

$$= 3^2 \cdot 2 - 1^2 \cdot 2 = 9 \cdot 2 = 18$$

12-18 (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C .

12. $\mathbf{F}(x, y) = x^2 \mathbf{i} + y^2 \mathbf{j}$,

C is the arc of the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$

13. $\mathbf{F}(x, y) = xy^2 \mathbf{i} + x^2y \mathbf{j}$,

$C: \mathbf{r}(t) = \langle t + \sin \frac{1}{2}\pi t, t + \cos \frac{1}{2}\pi t \rangle, 0 \leq t \leq 1$

$$\bar{\mathbf{F}} = \nabla f = \langle x^2, y^2 \rangle = \langle f_x, f_y \rangle$$

$$f_x = x^2 \rightarrow$$

$$f = \int x^2 dx + g(y) = \frac{1}{3}x^3 + g(y)$$

$$f = \int f_y dy + h(x)$$

$$= \int y^2 dy + h(x)$$

$$= \frac{1}{3}y^3 + h(x) \quad \text{SET } = \frac{1}{3}x^3 + g(y)$$

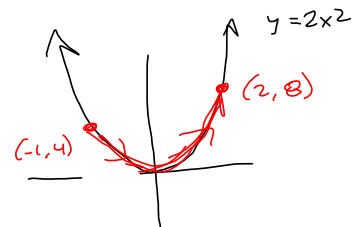
$$\rightarrow h(x) = \frac{1}{3}x^3 \text{ \& } g(y) = \frac{1}{3}y^3$$

$$f_y = 0 + g'(y) = y^2$$

$$g(y) = \frac{1}{3}y^3 + C$$

$$\rightarrow \left[f = \frac{1}{3}x^3 + \frac{1}{3}y^3 \right] + \text{Any constant}$$

0 is nice.



(b) $\int_C \bar{\mathbf{F}} \cdot d\bar{\mathbf{r}} = \int_C \nabla f \cdot d\bar{\mathbf{r}}$

$$= f(\bar{\mathbf{r}}(2, 8)) - f(\bar{\mathbf{r}}(-1, 2))$$

You do it with (-1, 2)...

$$= \frac{1}{3} \left[(2^3 + 8^3) - ((-1)^3 + 2^3) \right]$$

$$= \frac{1}{3} \left[2^3 + 8^3 + 1 - 2^3 \right]$$

$$= \frac{1}{3} \left[2^3 + 2^9 + 1 - 2^6 \right]$$

$$= \frac{1}{3} \left[2^3 (1 + 2^6 - 2^3) + 1 \right] = \frac{1}{3} \left[8 (57) + 1 \right]$$

$$= \frac{1}{3} \left[456 + 1 \right] = \frac{1}{3} \left[457 \right] = \frac{457}{3}$$

19-20 Show that the line integral is independent of path and evaluate the integral.

$$\frac{\partial}{\partial y} [\tan(y)] = \sec^2(y) \quad \checkmark$$

19. $\int_C \tan y \, dx + x \sec^2 y \, dy$, $\frac{\partial}{\partial x} [x \sec^2(y)] = \sec^2(y)$

C is any path from $(1, 0)$ to $(2, \pi/4)$

any path? $\vec{r}(t) = (1-t) \langle 1, 0 \rangle + t \langle 2, \frac{\pi}{4} \rangle$

$$\vec{F}(x(t), y(t)) = \langle \tan(y(t)), x(t) \sec^2(y(t)) \rangle = \nabla f \quad \text{for some } f$$

$$f_x = \tan(y)$$

$$f = \int \tan(y) \, dx + g(y) = x \tan(y) + g(y)$$

$$f = \int x \sec^2(x) \, dy + g(x) = xy \sec^2(x) + g(x)$$

$$\rightarrow f_y = x \sec^2(y) + g'(y) = x \sec^2(y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = \text{constant}$$

$$f = x \tan(y)$$

$$f(2, \frac{\pi}{4}) - f(1, 0) = 2 \tan \frac{\pi}{4} - 1 \tan(0) = \boxed{2}$$

$$d\vec{r} = \langle x'(t), y'(t) \rangle dt$$

$$\vec{F} \circ d\vec{r} = \langle \tan(y(t)), x(t) \sec^2(y(t)) \rangle \cdot \langle x'(t), y'(t) \rangle dt$$

$$= \tan(y(t)) x'(t) dt + x(t) \sec^2(y(t)) \underline{y'(t) dt}$$

$$= \tan(y) dx + x \sec^2(y) dy$$

21-22 Find the work done by the force field \mathbf{F} in moving an object from P to Q .

21. $\mathbf{F}(x, y) = 2y^{3/2}\mathbf{i} + 3x\sqrt{y}\mathbf{j}$; $P(1, 1)$, $Q(2, 4)$

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

\mathbf{F} conservative?

$$P_y = 3y^{1/2} \stackrel{?}{=} Q_x = 3\sqrt{y} = 3y^{1/2} \text{ Yes.}$$

Use the theorem.

$$f_x = 2y^{3/2} \Rightarrow f = 2xy^{3/2} + g(y)$$

$$\Rightarrow f_y = 3xy^{1/2} + g'(y) = 3x\sqrt{y} \Rightarrow$$

$$0 \stackrel{\text{SET}}{=} g'(y) = C \in \mathbb{R} \Rightarrow$$

$$f(x, y) = 2xy^{3/2}$$

$$f(2, 4) - f(1, 1) = 2(2)(4)^{3/2} - 2(1)(1)^{3/2}$$

$$= 2 \cdot 2 \cdot 2^3 - 2 = 32 - 2 = 30 = \text{Work}$$

$$\begin{aligned} \mathbf{r} &= (1-t)\langle 1, 1 \rangle + t\langle 2, 4 \rangle \\ &= \langle 1-t, 1-t \rangle + \langle 2t, 4t \rangle \\ &= \langle 1+t, 1+3t \rangle \\ &= \langle t+1, 3t+1 \rangle \end{aligned}$$

