

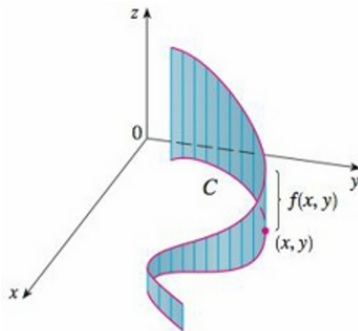
2 DEFINITION If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$

if this limit exists.

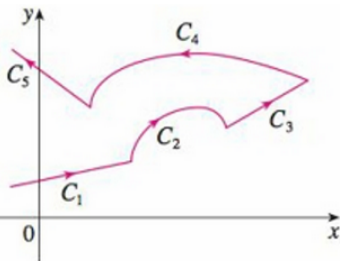
line integral with respect to arc length.

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$



One way of thinking line integrals is the area of the "fence."

This theory holds up for smooth curves C , and we can extend it to any curve that is piecewise smooth, by applying the ideas to the smooth pieces, separately, and then adding up the resulting integrals.



$$\int_C f(x, y) ds = \int_{C_1} f(x, y) ds + \int_{C_2} f(x, y) ds + \dots + \int_{C_n} f(x, y) ds$$

line integrals of f along C with respect to x and y :

$$\int_C f(x, y) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta x_i$$

$$\int_C f(x, y) dy = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta y_i$$

Notation that I dislike, but which is standard:

$$\int_C P(x, y) dx + \int_C Q(x, y) dy = \int_C P(x, y) dx + Q(x, y) dy$$

Integrating over a line segment is easy as pie in vector notation:

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$

Generally speaking, line (curve) integrals depend on the path taken:

We will be characterizing conditions where the integrals are INDEPENDENT of path, and it will be a combination of the domain in which the curve C is embedded (no holes) and the integrand, itself (conservative).

EXAMPLE 4 Evaluate $\int_C y^2 dx + x dy$, where (a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$ and (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$. (See Figure 7.)

In general, a given parametrization $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, determines an **orientation** of a curve C , with the positive direction corresponding to increasing values of the parameter t .

and the term

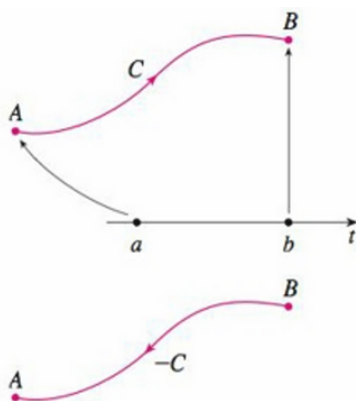
If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation (from initial point B to terminal point A in Figure 8), then we have

$$\int_{-C} f(x, y) dx = -\int_C f(x, y) dx \quad \int_{-C} f(x, y) dy = -\int_C f(x, y) dy$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y) ds = \int_C f(x, y) ds$$

This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of C .



LINE INTEGRALS IN SPACE

$$\int_C f(x, y, z) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta s_i$$

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

$$\int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt$$

And Arc Length looks like this: $\int_C ds = \int_a^b |\mathbf{r}'(t)| dt = L$

More 3-D versions of line integrals:

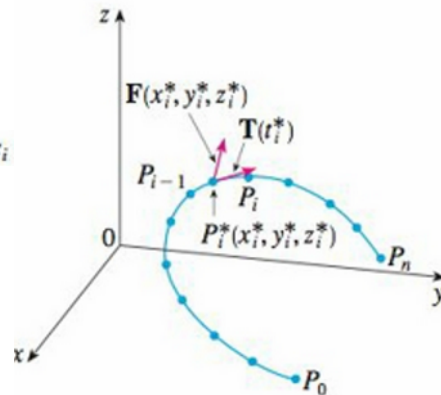
$$\int_C f(x, y, z) dz = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta z_i = \int_a^b f(x(t), y(t), z(t)) z'(t) dt$$

$$\int_C P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz$$

LINE INTEGRALS OF VECTOR FIELDS

Recall from Section 6.4 that the work done by a variable force $f(x)$ in moving a particle from a to b along the x -axis is $W = \int_a^b f(x) dx$. Then in Section 13.3 we found that the work done by a constant force \mathbf{F} in moving an object from a point P to another point Q in space is $W = \mathbf{F} \cdot \mathbf{D}$, where $\mathbf{D} = \overrightarrow{PQ}$ is the displacement vector.

We divide C into subarcs $P_{i-1}P_i$ with lengths Δs_i



If Δs_i is small, then as the particle moves from P_{i-1} to P_i along the curve, it proceeds approximately in the direction of $\mathbf{T}(t_i^*)$, the unit tangent vector at P_i^* .

Thus the work done by the force \mathbf{F} in moving the particle from P_{i-1} to P_i is approximately

$$\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \mathbf{T}(t_i^*)] = [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)] \Delta s_i$$

Add 'em all up:

$$\sum_{i=1}^n [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)] \Delta s_i$$

Pass to the limit as the maximum length of the subarcs approaches zero.

The work W done by the force field \mathbf{F}

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) ds = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

Re-stating in vector notation gives a nice, pretty expression we all saw coming:

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)| dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

This integral is often abbreviated as $\int_C \mathbf{F} \cdot d\mathbf{r}$ and occurs in other areas of physics as well. Therefore we make the following definition for the line integral of *any* continuous vector field.

13 DEFINITION Let \mathbf{F} be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of \mathbf{F} along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

That notation I hate arises quite naturally from this integral, when you remind yourself what things are:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_a^b (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}) dt \\ &= \int_a^b [P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)] dt \\ \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C P dx + Q dy + R dz \quad \text{where } \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}\end{aligned}$$

Thus we see a connection between vector representations and scalar representations.

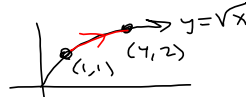
1-16 Evaluate the line integral, where C is the given curve.

5. $\int_C (x^2 y^3 - \sqrt{x}) dy,$

C is the arc of the curve $y = \sqrt{x}$ from $(1, 1)$ to $(4, 2)$

Let $x = \text{parameter}$

$$\begin{aligned} x^2 y^3 - \sqrt{x} \\ = x^2 (\sqrt{x})^3 - \sqrt{x} \\ = x^{\frac{7}{2}} - x^{\frac{1}{2}} \end{aligned}$$



$$\begin{aligned} dy &= y'(t) dt = y'(x) dx = \frac{1}{2} x^{-\frac{1}{2}} dx \\ y &= x^{\frac{1}{2}} \Rightarrow y' = \frac{1}{2} x^{-\frac{1}{2}} \end{aligned}$$

$$\int_1^4 (x^{\frac{7}{2}} - x^{\frac{1}{2}}) \left(\frac{1}{2} x^{-\frac{1}{2}} \right) dx$$

$$= \int_1^4 \left(\frac{1}{2} x^3 - \frac{1}{2} \right) dx = \left[\frac{x^4}{8} - \frac{1}{2} x \right]_1^4 = \frac{32}{8} - \frac{4}{2} - \left(\frac{1}{8} - \frac{1}{2} \right)$$

$$= 32 - 2 + \frac{3}{8} = 30 + \frac{3}{8} = \frac{240 + 3}{8} = \boxed{\frac{243}{8}}$$

II. $\int_C x e^{yz} ds,$

C is the line segment from $(0, 0, 0)$ to $(1, 2, 3)$

$$(1-t)\vec{r}_0 + t\vec{r}_1 = (1-t)\langle 0, 0, 0 \rangle + t\langle 1, 2, 3 \rangle = \langle t, 2t, 3t \rangle = \vec{r}(t)$$

$$ds = |\vec{r}'(t)| dt = \langle x(t), y(t), z(t) \rangle$$

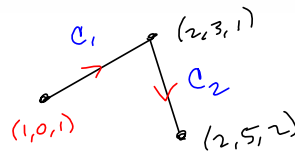
$$\vec{r}'(t) = \langle 1, 2, 3 \rangle \Rightarrow |\vec{r}'| = \sqrt{1+4+9} = \sqrt{14}$$

$$\int_0^1 (x(t) e^{y(t)z(t)}) \sqrt{14} dt = \int_0^1 t e^{2t \cdot 3t} \sqrt{14} dt$$

$$= \sqrt{14} \int_0^1 t e^{6t^2} dt = \frac{\sqrt{14}}{12} \int_0^1 e^{6t^2} \cdot 12t dt = \frac{\sqrt{14}}{12} \left[e^{6t^2} \right]_0^1$$

$$= \boxed{\frac{\sqrt{14}}{12} [e^6 - 1]}$$

15. $\int_C (x + yz) dx + 2x dy + xyz dz$, C consists of line segments from $(1, 0, 1)$ to $(2, 3, 1)$ and from $(2, 3, 1)$ to $(2, 5, 2)$



$$= \int_{C_1} (x + yz) dx + 2x dy + xyz dz$$

$dx = x'(t) dt$

$$+ \int_{C_2} (x + yz) dx + 2x dy + xyz dz$$

$$= \int_0^1 ((t+1) + 3t) dt + 2(t+1)3 dt + (t+1)(3t) \cdot 0 dt$$

$$+ \int_0^1 ((2+1) + (2+3)(t+1)) dt$$

$$+ 2(2(2t+3)) \cdot 2 dt + (2)(2t+3)(t+1) dt$$

$$= \int_0^1 (6t^2 + 41t + 37) dt$$

$$= \left[2t^3 + \frac{41}{2}t^2 + 37t \right]_0^1$$

$$= 2 + \frac{41}{2} + 37 = 39 + \frac{41}{2} = \frac{78+41}{2}$$

$$= \boxed{\frac{119}{2}}$$

$$C_1: (1-t) \langle 1, 0, 1 \rangle + t \langle 2, 3, 1 \rangle$$

$$= \langle 1-t, 0, 1-t \rangle + \langle 2t, 3t, t \rangle$$

$$= \langle t+1, 3t, 1 \rangle \rightarrow \vec{r}' = \langle 1, 3, 0 \rangle$$

$= \langle x', y', z' \rangle$

$$C_2: (1-t) \langle 2, 3, 1 \rangle + t \langle 2, 5, 2 \rangle$$

$$= \langle 2-2t, 3-3t, 1-t \rangle + \langle 2t, 5t, 2t \rangle$$

$$= \langle 2, 2t+3, t+1 \rangle = \langle x, y, z \rangle$$

$$\vec{r}'(t) = \langle 0, 2, 1 \rangle$$

$$= \langle x', y', z' \rangle$$

$$(2t+3)(t+1) = 2t^2 + 5t + 1$$

$$4t + 1 + 6t + 6$$

$$+ 2t^2 + 5t + 1 + 3 + 16t + 24$$

$$4t^2 + 10t + 2$$

$$= 6t^2 + 41t + 37$$

23–26 Use a calculator or CAS to evaluate the line integral correct to four decimal places.

23. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = xy \mathbf{i} + \sin y \mathbf{j}$ and

$$\mathbf{r}(t) = e^t \mathbf{i} + e^{-t^2} \mathbf{j}, \quad 1 \leq t \leq 2 \quad \mathbf{F} = \langle e^t, e^{-t^2} \rangle$$

$$\Rightarrow \mathbf{r}'(t) = \langle e^t, -2te^{-t^2} \rangle$$

$$\begin{aligned} \Rightarrow \mathbf{F} \cdot d\mathbf{r} &= xy e^t - 2 \sin y t e^{-t^2} \\ &= e^t e^{-t^2} e^t - 2 \sin(e^{-t^2}) t e^{-t^2} \\ &= e^{2t-t^2} - 2te^{-t^2} \sin(e^{-t^2}) \\ &\int_1^2 (e^{2t-t^2} - 2te^{-t^2} \sin(e^{-t^2})) dt \end{aligned}$$

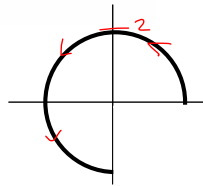
$$\int_1^2 (e^{2t-t^2} - 2te^{-t^2} \sin(e^{-t^2})) dt = \cos(e^{-1}) - \cos(e^{-4}) + \frac{\sqrt{\pi} e \operatorname{erf}(1)}{2} \approx 1.963338271$$

27-28 Use a graph of the vector field F and the curve C to guess whether the line integral of F over C is positive, negative, or zero. Then evaluate the line integral.

Guess: Net Positive

27. $F(x, y) = (x - y) \mathbf{i} + xy \mathbf{j}, = \langle x-y, xy \rangle$

C is the arc of the circle $x^2 + y^2 = 4$ traversed counter-clockwise from $(2, 0)$ to $(0, -2)$



$$\int_C \mathbf{F} \cdot d\mathbf{r} \quad \begin{matrix} x = 2 \cos \theta \\ y = 2 \sin \theta \end{matrix}$$

$$\mathbf{r}(\theta) = \langle 2 \cos \theta, 2 \sin \theta \rangle$$

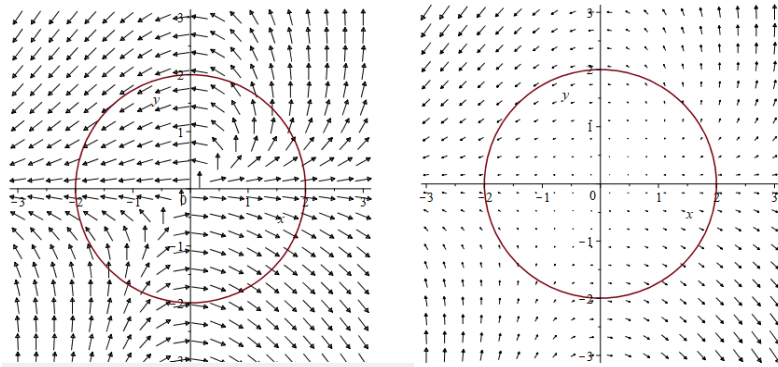
$$\mathbf{r}'(\theta) = \langle -2 \sin \theta, 2 \cos \theta \rangle$$

$$\int_0^{3\pi/2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{3\pi/2} \langle 2 \cos \theta - 2 \sin \theta, 4 \sin \theta \cos \theta \rangle \cdot \langle -2 \sin \theta, 2 \cos \theta \rangle d\theta$$

$$= \int_0^{3\pi/2} (-4 \sin \theta \cos \theta + 4 \sin^2 \theta + 8 \cos^2 \theta \sin \theta) d\theta$$

$$= \left[2 \cos^2 \theta + 2\theta + \sin(2\theta) - \frac{8}{3} \cos^3 \theta \right]_0^{3\pi/2}$$

$$= 2 \cdot 0 + 2 \cdot \frac{3\pi}{2} + 0 - \frac{8}{3} \cdot 0 - \left[2 \cdot 1 + 2 \cdot 0 + 0 - \frac{8}{3} \cdot 1^3 \right] = 3\pi - \left[2 - \frac{8}{3} \right] = 3\pi + \frac{2}{3}$$



The field plot on the right was obtained by the following commands:

```
with(plots):
myfield := fieldplot([x - y, x*y], x=-3..3, y=-3..3, arrows=slim):%
mycircle := implicitplot(x^2 + y^2 = 4, x=-3..3, y=-3..3):%
display([myfield, mycircle])
```

The field plot on the left was obtained by the following commands:

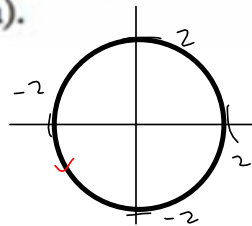
```
myfield := fieldplot([x - y, x*y], x=-3..3, y=-3..3, arrows=slim, fieldstrength=fixed(.9)):%
mycircle := implicitplot(x^2 + y^2 = 4, x=-3..3, y=-3..3):%
display([myfield, mycircle])
```

By fixing the "field strength" option, we obtain a better sense of which direction things are pointing.

By NOT fixing it, we get a true measure of the intensity of the field.

You only need one "with(plots):" for the whole thing. Also mycircle was defined the same way, both times, so it didn't need to be defined a second time. The main thing, here, is the **fieldstrength** option.

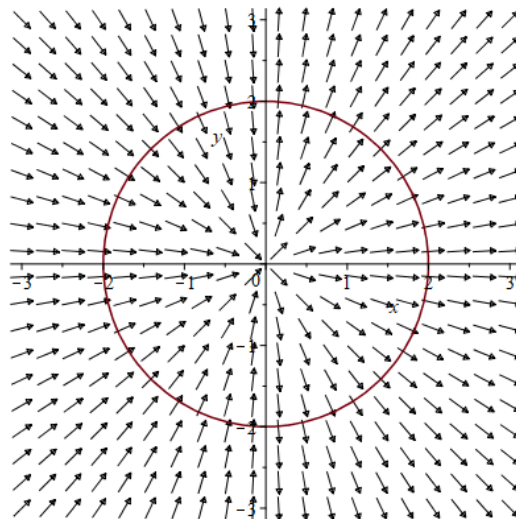
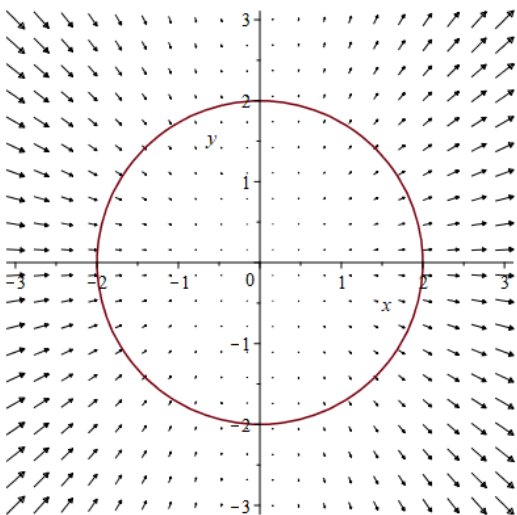
32. (a) Find the work done by the force field $\mathbf{F}(x, y) = x^2 \mathbf{i} + xy \mathbf{j}$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented in the counterclockwise direction.
- (b) Use a computer algebra system to graph the force field and circle on the same screen. Use the graph to explain your answer to part (a).



$$\begin{aligned}\bar{r}(t) &= \langle 2 \cos \theta, 2 \sin \theta \rangle \\ \bar{r}'(t) &= \langle -2 \sin \theta, 2 \cos \theta \rangle\end{aligned}$$

$$\begin{aligned}& \int_0^{2\pi} \langle 4 \cos^2 \theta, 4 \sin \theta \cos \theta \rangle \cdot \langle -2 \sin \theta, 2 \cos \theta \rangle d\theta \\ &= \int_0^{2\pi} (-8 \cos^2 \theta \sin \theta + 8 \sin \theta \cos^2 \theta) d\theta = 0\end{aligned}$$

The field on the left is the actual. The one on the right is where I made all the vectors the same length, so that I could see the direction everything was pointing. A good idea to do both.



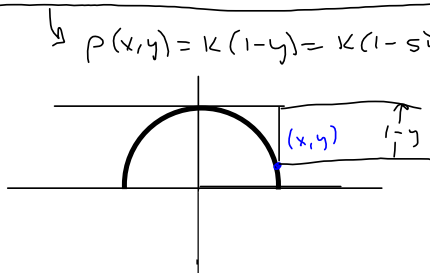
37. If a wire with linear density $\rho(x, y)$ lies along a plane curve C , its **moments of inertia** about the x - and y -axes are defined as

$$I_x = \int_C y^2 \rho(x, y) ds \quad I_y = \int_C x^2 \rho(x, y) ds \quad ds = |\vec{r}'(\theta)| d\theta = d\theta$$

$$\vec{r} = \langle \cos \theta, \sin \theta \rangle$$

Find the moments of inertia for the wire in Example 3. $\vec{r}' = \langle -\sin \theta, \cos \theta \rangle$

EXAMPLE 3 A wire takes the shape of the semicircle $x^2 + y^2 = 1, y \geq 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.



$\rho(x, y) = k(1-y) = k(1-\sin \theta)$

$$I_x = k \int_0^\pi \sin^2 \theta (1 - \sin \theta) d\theta = k \left(\frac{\pi}{2} - \frac{1}{3} \right)$$

$\sin^2 \theta = \frac{1}{2}(1 - \cos(2\theta))$

$\sin^3 \theta = (1 - \cos^2 \theta) \sin \theta$

$$I_y = k \int_0^\pi \cos^2 \theta (1 - \sin \theta) d\theta = \left(\frac{\pi}{2} - \frac{2}{3} \right) k$$

43. A 160-lb man carries a 25-lb can of paint up a helical staircase that encircles a silo with a radius of 20 ft. If the silo is 90 ft high and the man makes exactly three complete revolutions, how much work is done by the man against gravity in climbing to the top?

$$\vec{F} = \langle 0, 0, 185 \rangle$$

$$\vec{r} = \langle x(\theta), y(\theta), z(\theta) \rangle$$

$$= \langle 20 \cos \theta, 20 \sin \theta, ? \rangle$$

$$z(0) = 0, z(6\pi) = 90$$

z increased @ a fixed rate.

$\therefore z(\theta)$ is LINEAR!

$$\text{pts } (\theta_1, z_1) = (0, 0)$$

$$(\theta_2, z_2) = (6\pi, 90)$$

$$m = \frac{90}{6\pi} = \frac{30}{2\pi} = \frac{15}{\pi}$$

$$z = \frac{15}{\pi}(\theta - 0) + 0$$

$$z = \frac{15}{\pi}\theta$$

$$\rightarrow \langle 20 \cos \theta, 20 \sin \theta, \frac{15}{\pi}\theta \rangle$$

$$= \vec{r}(\theta). \implies \vec{r}'(\theta) = \langle -20 \sin \theta, 20 \cos \theta, \frac{15}{\pi} \rangle$$

$$\text{WORK} = \int_C \vec{F} \cdot d\vec{r} = \int_C \langle 0, 0, 185 \rangle \cdot \langle -20 \sin \theta, 20 \cos \theta, \frac{15}{\pi} \rangle d\theta$$

$$= \int_0^{6\pi} \frac{15 \cdot 185}{\pi} d\theta = \int_0^{6\pi} \frac{2775}{\pi} d\theta$$

$$= \frac{2775}{\pi} \theta \Big|_0^{6\pi} = \frac{2775}{\pi} \cdot \underset{\text{ft}}{6\pi} = 16650 \text{ ft-lbs.}$$

$$\text{Note: } \underset{\text{ft}}{(90)} \underset{\text{lbs}}{(185)} = 16650 \text{ ft-lbs. !}$$

3 revs

$$3 \cdot 2\pi = 6\pi$$

