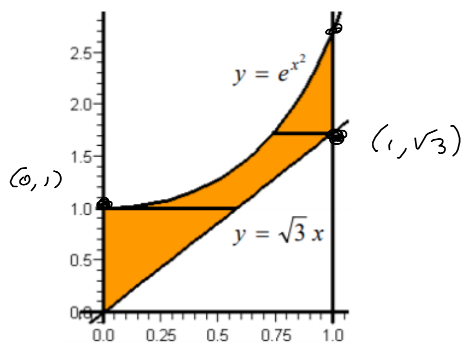
1. (15 pts) Evaluate the iterated integral $\displaystyle\int_0^1 \int_{\sqrt{3}x}^{e^{x^2}} 8xy\,dy\,dx$. A sketch of the Type I region $\boldsymbol{R}$ over which this integral is taken is given on the right, with some additional information you might find helpful for #2.

$$\int_0^1 \left[4xy^2\right]_{\sqrt{3}x}^{e^{x^2}} dx = \int_0^1 4x\left(\left[e^{x^2}\right]^2 - 3x^2\right) dx$$

$$= \int_0^1 \left(4xe^{2x^2} - 12x^3\right) dx = e^{2x^2}\Big]_0^1 - 3x^4\Big]_0^1$$

$$= e^2 - e^0 - (3) = e^2 - 1 - 4 = \boxed{e^2 - 4}$$

2. (15 pts) Re-write the integral in #1, to give you the volume under $f(x, y) = 4xy$, by viewing $\boldsymbol{R}$ as a Type II region. This will require 3 different iterated integrals. The extra horizontal lines are meant to be a hint. *Do not evaluate!!!*




$e^{0^2} = e^0 = 1$

$\sqrt{3}(1) = \sqrt{3}$

$e^{1^2} = e^1 = e$

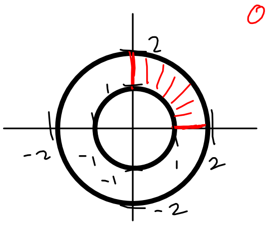
$y = e^{x^2} \Rightarrow \ln(y) = x^2 \Rightarrow x = \pm\sqrt{\ln(y)} = +\sqrt{\ln(y)}$

$y = \sqrt{3}\,x$

$x = \frac{1}{\sqrt{3}}y$

$$\int_0^1 \int_0^{\frac{1}{\sqrt{3}}y} 8xy\,dx\,dy + \int_1^{\sqrt{3}} \int_{\sqrt{\ln(y)}}^{\frac{1}{\sqrt{3}}y} 8xy\,dx\,dy + \int_{\sqrt{3}}^{e} \int_{\sqrt{\ln(y)}}^{1} 8xy\,dx\,dy$$

3. (15 pts) Evaluate the iterated integral $\iint_D (x+y)\,dA$, where $D$ is the region in the $1^{st}$ quadrant, between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$, by converting to polar coordinates.




$0 \le \theta \le \frac{\pi}{2}$

$1 \le r \le 2$

$x = r\cos\theta$

$y = r\sin\theta$

$$\int_0^{\frac{\pi}{2}} \int_1^2 r(\cos\theta + \sin\theta) \cdot r\,dr\,d\theta$$

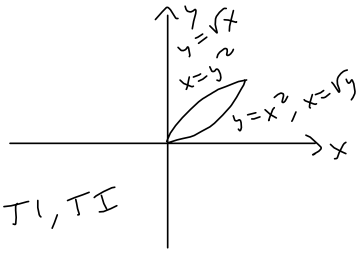
$$= \int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta)\,d\theta \int_1^2 r^2\,dr$$

$$= \left(\sin\theta - \cos\theta\Big]_0^{\frac{\pi}{2}}\right)\left(\frac{1}{3}r^3\Big]_1^2\right) = \Big((1-0)-(0-1)\Big)\left(\frac{8-1}{3}\right)$$

$$= \boxed{\frac{14}{3}}$$

4. (15 pts) Evaluate the triple integral $\iiint_{\mathcal{E}} xy\,dV$, where $\mathcal{E}$ is the solid in the first octant bounded by the parabolic cylinders $y = x^2, x = y^2$ and the planes $z = 0$ and $z = x + y$

$x + y - z = 0$ Plane

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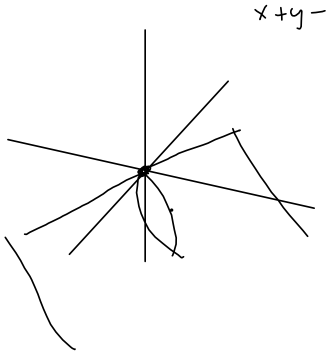
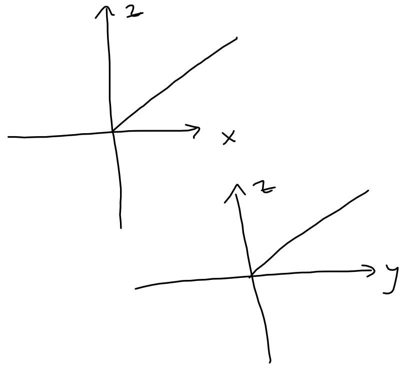





$$\int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xy \;dz\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \Big[xyz\Big]_0^{x+1} dy\,dx = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2y + xy)\,dy\,dx = \int_0^1 \Big[\tfrac{1}{2}x^2y^2 + \tfrac{1}{2}xy^2\Big]_{x^2}^{\sqrt{x}} dx$$

$xy(x+1) = x^2 + xy$

$$= \int_0^1 \Big[\Big(\tfrac{1}{2}x^2\cdot x + \tfrac{1}{2}x\cdot x\Big) - \Big(\tfrac{1}{2}x^2\cdot x^4 + \tfrac{1}{2}x\cdot x^4\Big)\Big] dx$$

$$= \int_0^1 \Big[\tfrac{1}{2}x^3 + \tfrac{1}{2}x^2 - \tfrac{1}{2}x^6 - \tfrac{1}{2}x^5\Big] dy$$

$$= \Big[\tfrac{1}{8}x^4 + \tfrac{1}{6}x^3 - \tfrac{1}{14}x^7 - \tfrac{1}{12}x^6\Big]_0^1 = \tfrac{1}{8} + \tfrac{1}{6} - \tfrac{1}{14} - \tfrac{1}{12}$$

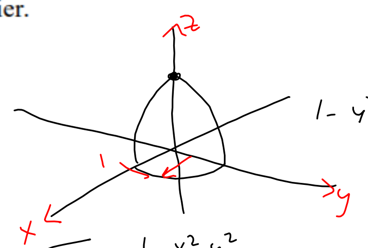
$$= \tfrac{1}{8} + \tfrac{1}{12} - \tfrac{1}{14} = \frac{21+14-24}{168} = \frac{35-24}{168}$$

$2\cdot 7\cdot 2\cdot 3\cdot 2$

$$\begin{array}{r} 56 \\ \times\; 3 \\ \hline 168 \end{array}$$

$= \frac{11}{168}$ Something Wrong

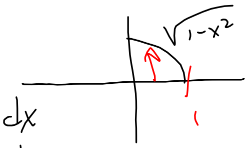
5. (15 pts) Evaluate the triple integral $\iiint_{\mathcal{E}} \left(x^3 + xy^2\right) dV$, where $\mathcal{E}$ is the solid in the first octant that lies beneath the paraboloid $z = 1 - x^2 - y^2$. Hint: Converting to Cylindrical Coordinates after you've found the triple integrals limits of integration will make evaluation easier.




Intersection w/ xy-plane: $z=0$

$1 - y^2 - x^2 = 0$

$x^2 + y^2 = 1$




$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} \left(x^3 + xy^2\right) \underbrace{dz\,dy\,dx}_{dV}$$

$1 - x^2 - y^2$
$= 1 - (x^2+y^2) = 1 - r^2$

To cylindrical coordinates

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{1-r^2} \left(r^3\cos^3\theta + r\cos\theta \cdot r^2\sin^2\theta\right) \underbrace{r\,dz\,d\theta\,dr}_{dV}$$

Scratch: $r^4\cos\theta\left[\cos^2\theta + \sin^2\theta\right]$

$$= \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{1-r^2} r^4\cos\theta\,dz\,d\theta\,dr$$

$$= \int_0^1 \int_0^{\frac{\pi}{2}} (1-r^2)\,r^4\cos\theta\,d\theta\,dr = \int_0^1 (r^4 - r^6)\,dr \int_0^{\frac{\pi}{2}} \cos\theta\,d\theta$$

$$= \left[\frac{r^5}{5} - \frac{r^7}{7}\right]_0^1 \left[+\sin\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{5} - \frac{1}{7} = \frac{7-5}{35} = \boxed{\frac{2}{35}}$$

6. (15 pts) Compute the Jacobian for the transformation $u = x + y, v = 2x + 3y$.

We need $x = x(u,v)$ & $y = y(u,v)$ $\quad \dfrac{\partial(x,y)}{\partial(u,v)}$

we have $u = u(x,y)$ & $v = v(x,y)$

We're <u>given</u> $T^{-1}$. Want $T$.

xy-plane $R \xleftarrow{T} S$ uv-plane

$(x,y) \xrightarrow{T^{-1}} (x+y, 2x+3y) = (u,v)$

Trick: $|T| = \dfrac{1}{|T^{-1}|}$

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3-2 = 1 = |T^{-1}|$$

$$\Longrightarrow |T| = \frac{1}{1} = 1$$

$$T^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \Longrightarrow T = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

$$|T| = \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} = \boxed{1 = \frac{\partial(x,y)}{\partial(u,v)}}$$

This says

$x = 3u - v$

$y = -2u + v$

$x_u = 3 \quad x_v = -1$

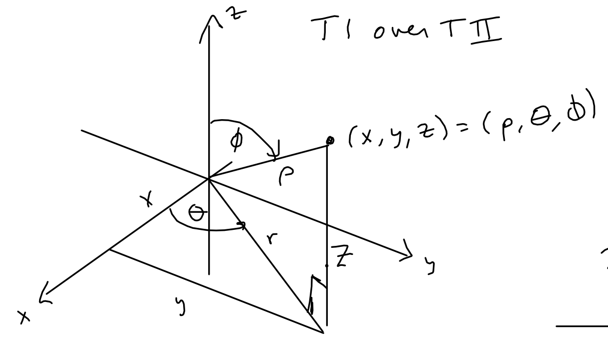
$y_u = -2 \quad y_v = 1 \quad \rightsquigarrow \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$ See?

7. (5 pts) Use spherical coordinates to evaluate

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$$

Re-writing in spherical coordinates is a legit question for the sit-down portion of the test. Evaluating it is probably not.

TI over TII

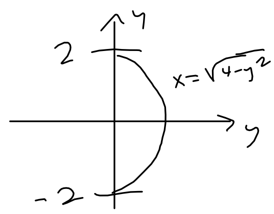


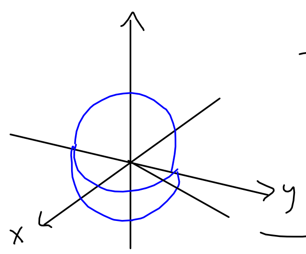

$z = \pm\sqrt{4-x^2-y^2}$

$z^2 = 4-x^2-y^2$

$x^2+y^2+z^2=4$

sphere of radius $r=2$




$0 \le \phi \le \pi$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$0 \le \rho \le 2$

$y^2 = (\rho\sin\phi\sin\theta)^2$

$\sqrt{x^2+y^2+z^2} = \rho$

Abs value of Jacobian

$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right|$

$$\boxed{\int_0^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2} \rho^2\sin^2\phi\sin^2\theta\cdot\rho\cdot\boxed{\rho^2\sin\phi}\,d\rho\,d\theta\,d\phi}$$

Derive the dV conversion for change of variables

This is extra!

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{array}{ccc} + & - & + \end{array}$$
$$\begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$
$$\begin{array}{ccc} + & - & + \end{array}$$

OR do $\bar{u}\cdot(\bar{v}\times\bar{w})$, where

$\bar{u} =$ top row

$\bar{v} =$ middle

$\bar{w} =$ bottom

$$= \cos\phi\left(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta\right)$$
$$- 0$$
$$- \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$$
$$= -\cos\phi\left(\rho^2\sin\phi\cos\phi\left(\sin^2\theta+\cos^2\theta\right)\right)$$
$$- \rho^2\sin\phi\left(\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right)\right) = -\rho^2\sin^3\phi\cos^2\phi$$

Missing power on video.

Missing "="? $- \rho^2\sin\phi\sin^2\phi$

$$= -\rho^2\sin\phi\left(\cos^2\phi+\sin^2\phi\right)$$
$$= \boxed{-\rho^2\sin\phi} = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$$

in the integral, take its absolute value

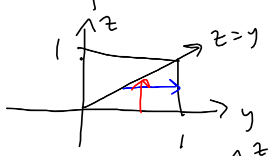
8. (5 pts) Give five other iterated integrals that are equal to $\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx$

Finding all 5 under a time control isn't the best idea. But specifying a couple different orderings of integration is reasonable, for instance, one re-ordering might be given by the following:

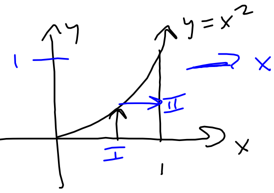
$$\int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} f(x,y,z)\ dy\,dx\,dz$$

I can see asking for 1 or 2 specific re-orderings.

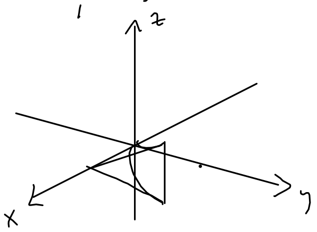
Type 1: from xy-plane $(z=0)$ to $z=y$ (plane)

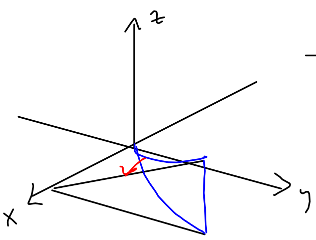


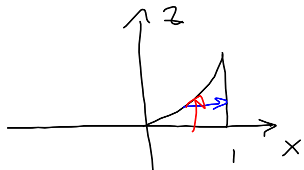

Type I




$\Rightarrow x=\sqrt{y}$




$z=y$ & $y=x^2$

$z=x^2$

$\Rightarrow x=\sqrt{z}$

$$\int_0^1\int_z^1\int_{\sqrt{y}}^1 dx\,dy\,dz$$

$$\int_0^1\int_0^y\int_{\sqrt{y}}^1 dx\,dz\,dy$$

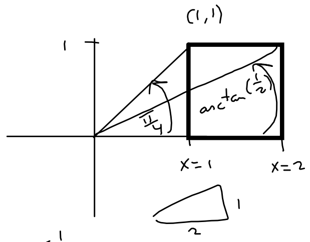
$$\int_0^1\int_{\sqrt{y}}^1\int_0^y dz\,dx\,dy$$

$$\int_0^1\int_{\sqrt{z}}^1\int_{x^2}^1 dy\,dx\,dz$$

$$\int_0^1\int_0^{x^2}\int_{x^2}^1 dy\,dz\,dx$$

9. (5 pts) Evaluate the integral in two ways:
   a. as written
   b. using polar coordinates

$$\int_0^1\int_1^2 \frac{x}{x^2+y^2}\,dx\,dy$$




VERY tough to evaluate in rectangular coordinates as it requires integration by parts, TWICE!

$$=\frac{1}{2}\int_0^1\int_1^2 \frac{2x}{x^2+y^2}\,dx\,dy = \frac{1}{2}\int_0^1 \ln(x^2+y^2)\Big]_1^2 dy$$

$u=x^2+y^2 \Rightarrow du = 2x\,dx$

$\int \frac{du}{u} = \ln|u|$

$$= \frac{1}{2}\int_0^1 \left(\ln(y^2+4) - \ln(y^2+1)\right) dy$$

Integration by parts... Ouch!

Convert:

$r$: outer: $x=2=r\cos\theta$

$\Rightarrow r = \frac{2}{\cos\theta} = 2\sec\theta$

$$\frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{1}{r}\cos\theta$$

Inner: $x=1 \Rightarrow r=\sec\theta$

$$\int_0^{\arctan(\frac{1}{2})}\int_{\sec\theta}^{2\sec\theta} \frac{1}{r}\cos\theta\, r\,dr\,d\theta + \int_{\arctan(\frac{1}{2})}^{\frac{\pi}{4}}\int_{\sec\theta}^{\csc\theta} \cos\theta\,dr\,d\theta$$

$y=1=r\sin\theta$

$r=\frac{1}{\sin\theta}=\csc\theta$

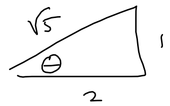
$$= \int_0^{\arctan(\frac{1}{2})}\int_{\sec\theta}^{2\sec\theta} \cos\theta\,dr\,d\theta + \int_{\arctan(\frac{1}{2})}^{\frac{\pi}{4}}\int_{\sec\theta}^{\csc\theta} \cos\theta\,dr\,d\theta$$

$$= \int_0^{\arctan(\frac{1}{2})} r\cos\theta\Big]_{\sec\theta}^{2\sec\theta} d\theta + \int_{\arctan(\frac{1}{2})}^{\frac{\pi}{4}} r\cos\theta\Big]_{\sec\theta}^{\csc\theta} d\theta$$

$$= \int_0^{\arctan(\frac{1}{2})} (\underbrace{2\sec\theta\cos\theta}_{=1} - \sec\theta\cos\theta)\,d\theta + \int_{\arctan(\frac{1}{2})}^{\frac{\pi}{4}} (\csc\theta\cos\theta - \sec\theta\cos\theta)\,d\theta$$

$$= \int_0^{\arctan(\frac{1}{2})} d\theta + \int_{\arctan(\frac{1}{2})}^{\frac{\pi}{4}} (\cot\theta - 1)\,d\theta$$

$$= \theta\Big]_0^{\arctan(\frac{1}{2})} + \Big[\ln|\sin\theta| - \theta\Big]_{\arctan(\frac{1}{2})}^{\frac{\pi}{4}}$$




$$= \arctan(\tfrac{1}{2}) + \ln|\sin\tfrac{\pi}{4}| - \tfrac{\pi}{4} - \left(\ln|\sin(\arctan(\tfrac{1}{2}))| - \arctan(\tfrac{1}{2})\right)$$

$$= \arctan(\tfrac{1}{2}) + \ln(\tfrac{1}{\sqrt{2}}) - \tfrac{\pi}{4} - \left(-\ln(\sqrt{5}) - \arctan(\tfrac{1}{2})\right)$$

$$= 2\arctan(\tfrac{1}{2}) - \ln(\sqrt{2}) - \tfrac{\pi}{4} + \ln(\sqrt{5})$$

$= 2\arctan(\frac{1}{2}) -$

$$\ln(\sqrt{2}) = \ln(2^{\frac{1}{2}}) = \tfrac{1}{2}\ln(2) = \frac{\ln(2)}{2}$$

$$\ln(\sqrt{5}) = \tfrac{1}{2}\ln(5)$$

$$\ln|u| = \int\frac{du}{u} = \int \frac{\cos\theta}{\sin\theta}\,d\theta = \ln|\sin\theta| + C$$

$u=\sin\theta$

$du=\cos\theta\,d\theta$

$$= \frac{1}{2}\int_0^1 \left(\ln(y^2+4) - \ln(y^2+1)\right) dy$$

$u = \ln(y^2+4)$

$du = \frac{2y}{y^2+4}\,dy$

$dv = dy$

$v = y$

$$= \frac{1}{2}\left[uv - \int v\,du\right] = \frac{1}{2}\left[y\ln(y^2+4) - \int_0^1 \frac{2y^2}{y^2+4}\,dy\right]$$

ouch! still hurts!