16.9 Change of Variable(s) in Multiple Integrals

Recall the Substitution Rule with the variables reversed:

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

$x = g(u)$ and $a = g(c)$, $b = g(d)$

$$\int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$$

$\int_c^d f(u)\,du = \int_a^b f(g(x))\,g'(x)\,dx$

$\int_0^2 \cos(u)\,du = \int_0^1 \cos(2x)\cdot 2\,dx$

$u = g(x) = 2x$

$du = g'(x)dx = 2dx$

$x = 0$

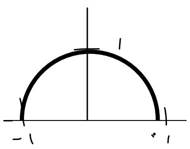
$u = 2x = 0 = g(0)$

$x = 1$

$u = 2\cdot 1 = 2 = g(1)$

Example: Trigonometric Substitution to simplify the integration process.

Find the area inside half a circle of radius $r = 1$.



$x^2 + y^2 = 1$

$y^2 = 1 - x^2$

$y = \pm\sqrt{1-x^2}$

$y = \sqrt{1-x^2}$ is the top

$\text{Area} = \int_{-1}^{1} \sqrt{1-x^2}\,dx$

$x = \cos\theta$

$dx = -\sin\theta\,d\theta$

$x = -1 = \cos\theta$

$\theta = \pi$

$x = 1 = \cos\theta$

$\theta = 0$

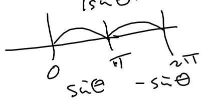
$\sqrt{1-x^2}$

$= \sqrt{1-(\cos\theta)^2}$

$= \sqrt{1-\cos^2\theta} = \sqrt{\sin^2\theta}$

$= |\sin\theta| = \sin\theta \ \forall\ \theta\ [0,\pi]$

$|\sin\theta|$



$\int_\pi^0 \sin\theta\,(-\sin\theta\,d\theta)$

$= -\int_\pi^0 \sin^2\theta\,d\theta$

$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

$= \int_0^\pi \frac{1-\cos(2\theta)}{2}\,d\theta$

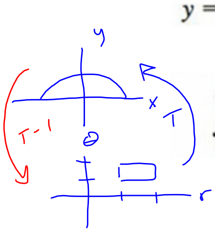
$= \int_0^\pi \frac{1}{2}\,d\theta - \frac{1}{2}\cdot\frac{1}{2}\int_0^\pi \cos(2\theta)\cdot 2\,d\theta$

$= \frac{1}{2}\pi - \frac{1}{4}\left[\sin 2\theta\right]_0^\pi = \frac{\pi}{2}$

We've already extended this to double integrals and polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$



$$\iint_R f(x, y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

is how we go from $dA = dx\,dy = dy\,dx$ to $r\,dr\,d\theta$

JACOBIAN IS THE BRIDGE

$x = r\cos\theta,\ y = r\sin\theta$

$T(r,\theta) = (x,y)$

The idea will be to Do a T^{-1}:

$T^{-1}(T(r,\theta)) = T^{-1}(x,y) = (r,\theta)$

$$T(u, v) = (x, y)$$

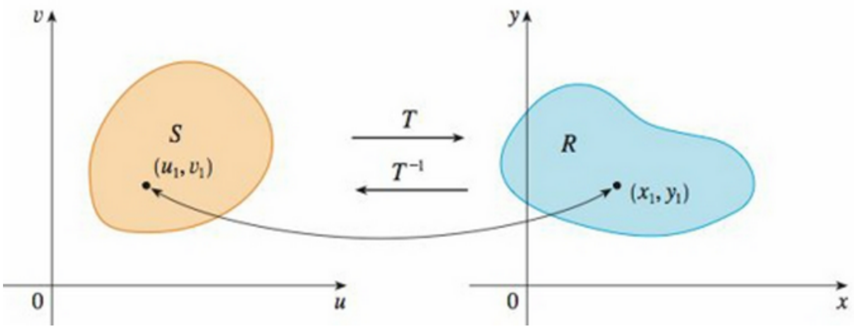
$x = r\cos\theta$ Dif6$^{\ell}$

$x = g(u, v)$ $\qquad$ $y = h(u, v)$

We usually assume that T is a C^1 **transformation**, which means that g and h have continuous first-order partial derivatives.

Transformations (Mappings) from one domain *onto* another.

If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1)



$u = G(x, y)$ $\qquad\qquad$ $v = H(x, y)$

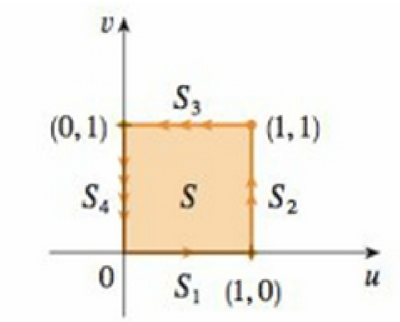
EXAMPLE 1 A transformation is defined by the equations

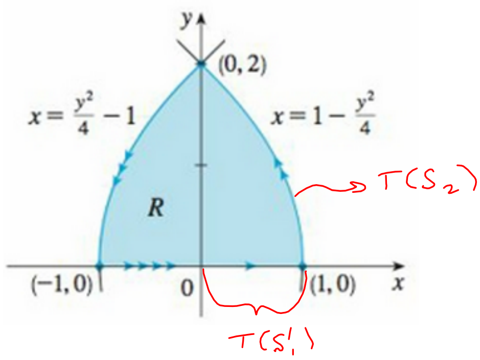
$T^{-1}(x,y) = (u^2 - v^2, 2uv)$ $\quad x = u^2 - v^2 \qquad y = 2uv$

Find the image of the square $S = \{(u, v) \mid 0 \leq u \leq 1,\ 0 \leq v \leq 1\}$

SOLUTION The transformation maps the boundary of S into the boundary of the image.

is reported as a fact, with no real proof, even though it is true.





$T(S_1)$: $v = 0$, $0 \leq u \leq 1$

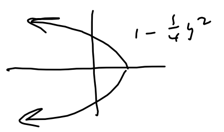
$x = u^2 - v^2 = u^2 \qquad y = 2uv = 0$

$\Rightarrow 0 \leq x \leq 1$

$T(S_2)$: $u = 1$, $0 \leq v \leq 1$

$x = 1^2 - v^2$, $y = 2v \Rightarrow v = \frac{1}{2}y$

$x = 1 - v^2 = 1 - (\frac{1}{2}y)^2 = 1 - \frac{1}{4}y^2$

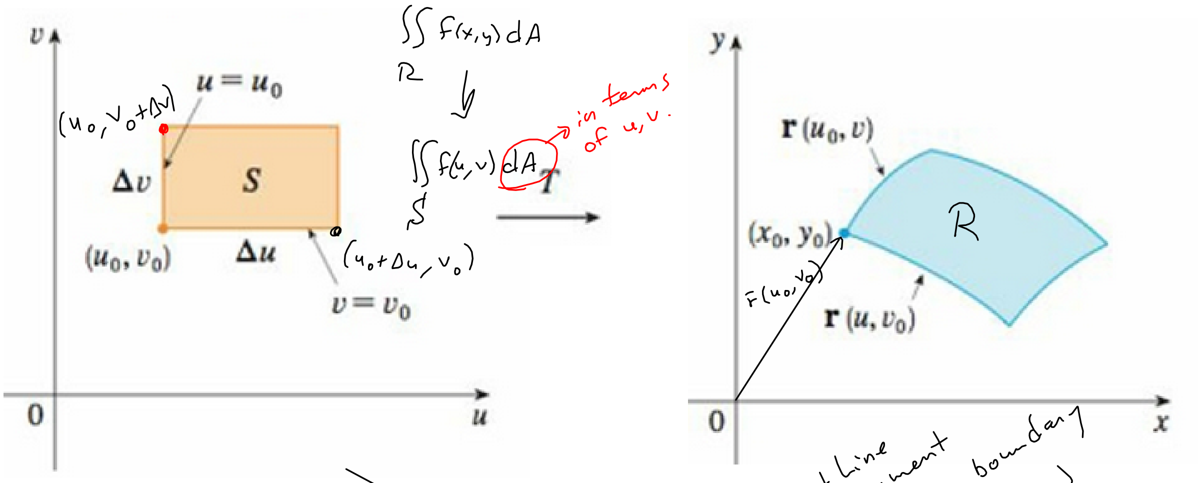


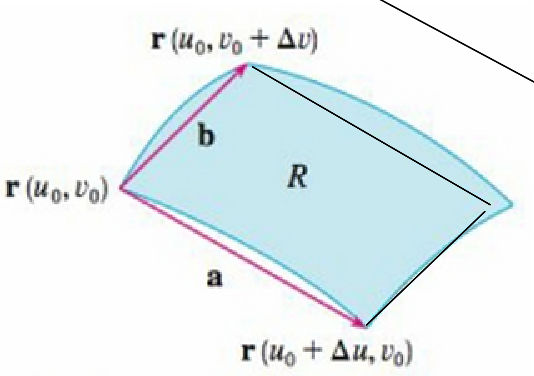
$0 \leq v \leq 1$

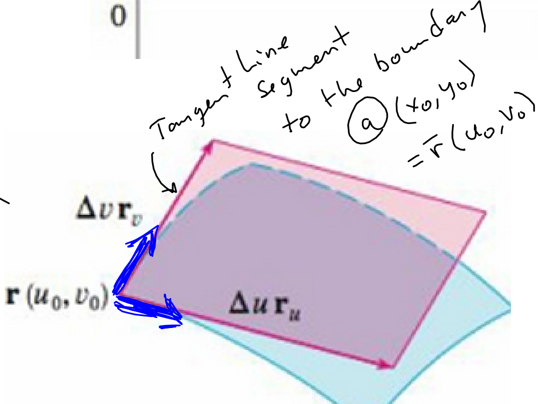
& $y = 2v \Rightarrow$

$0 \leq 2v \leq 2$

$0 \leq y \leq 2$







Area of parallelogram is $\|\bar{a} \times \bar{b}\|$

Derivation:

$$\Delta \bar{r} = \bar{b} = \bar{r}(u_0, v_0+\Delta v) - \bar{r}(u_0, v_0)$$

$$\Rightarrow \frac{\Delta \bar{r}}{\Delta v} = \frac{\bar{r}(u_0, v_0+\Delta v) - \bar{r}(u_0, v_0)}{\Delta v} \approx \frac{\partial \bar{r}}{\partial v} = \bar{r}_v$$

$$\Rightarrow \bar{b} = \bar{r}(u_0, v_0+\Delta v) - \bar{r}(u_0, v_0)$$

$$= \frac{\bar{r}(u_0, v_0+\Delta v) - \bar{r}(u_0, v_0)}{\Delta v} \cdot \Delta v$$

$$\approx \frac{\partial \bar{r}}{\partial v} \cdot \Delta v = \bar{r}_v \Delta v$$

$$\bar{a} = \frac{\partial \bar{r}}{\partial u} \cdot \Delta u = \bar{r}_u \Delta u$$

$$\Rightarrow \|\bar{a} \times \bar{b}\| \approx \|(\bar{r}_u \Delta u) \times (\bar{r}_v \Delta v)\| = \|\bar{r}_u \times \bar{r}_v\| \Delta u \Delta v = dA$$

The increment of area for the conversion!

Nov 9-10:55 AM

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Area of the region R using mapping T & domain S!

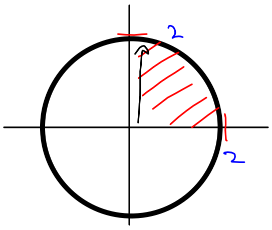
7 DEFINITION The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Double Integral in Polar Coordinates. Where did the $r\,dr\,d\theta$ come from?

We know, but here we do it in terms of the new machinery:

$$\int_0^1 \int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$$



$x = r\cos\theta$, $y = r\sin\theta$

$T(r\cos\theta, r\sin\theta) = (x, y)$

$\bar{r}(r,\theta) = \langle r\cos\theta, r\sin\theta \rangle$

$\bar{r}_r = \langle \cos\theta, \sin\theta \rangle$

$\bar{r}_\theta = \langle -r\sin\theta, r\cos\theta$

$\|\bar{r}_r \times \bar{r}_\theta\|$

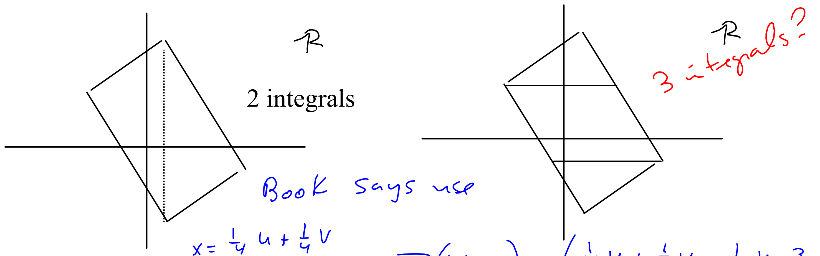
$$\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = |r\cos^2\theta + r\sin^2\theta|$$

$= |r| = r$ if $0 \le r$

It's the r in the $r\,dr\,d\theta$!

#12

$\iint_R (4x+8y)\,dA$, where R is the parallelogram with ~~corns~~ corners $(-1,3)$, $(1,-3)$, $(3,-1)$, $(1,5)$



Book says use

$x = \frac{1}{4}u + \frac{1}{4}v$

$y = \frac{1}{4}v - \frac{3}{4}u$

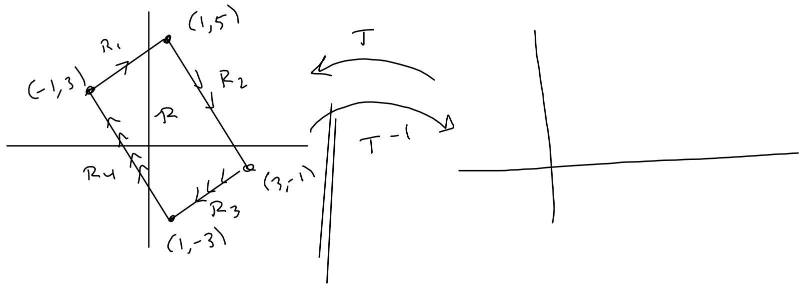
$T(u,v) = \left(\frac{1}{4}u + \frac{1}{4}v,\ \frac{1}{4}v - \frac{3}{4}u\right) = (x,y)$

$\bar{r}(u,v) = \left\langle \frac{1}{4}u + \frac{1}{4}v,\ \frac{1}{4}v - \frac{3}{4}u \right\rangle$

$\bar{r}_u = \left\langle \frac{1}{4}, -\frac{3}{4} \right\rangle$, $\bar{r}_v = \left\langle \frac{1}{4}, \frac{1}{4} \right\rangle$

$$\begin{vmatrix} \frac{1}{4} & -\frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{16} + \frac{3}{16} = \frac{4}{16} = \boxed{\frac{1}{4} = \frac{\partial(x,y)}{\partial(u,v)}}$$

$$4x + 8y = 4\left(\frac{1}{4}u + \frac{1}{4}v\right) + 8\left(\frac{1}{4}v - \frac{3}{4}u\right)$$
$$= u + v + 2v - 6u = -5u + 3v$$



To get T^{-1}, solve for u and v:

$x = \frac{1}{4}u + \frac{1}{4}v$

$y = \frac{1}{4}v - \frac{3}{4}u$

$$\begin{aligned} \tfrac{1}{4}u + \tfrac{1}{4}v &= x \\ -\tfrac{3}{4}u + \tfrac{1}{4}v &= y \end{aligned} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{4} & \frac{1}{4} & x \\ -\frac{3}{4} & \frac{1}{4} & y \end{array}\right]$$

Reduced Row-Echelon Form: $\sim \left[\begin{array}{cc|c} 1 & 0 & x-y \\ 0 & 1 & 3x+y \end{array}\right]$

$\Rightarrow$ $u = x - y$, $v = 3x + y$

$T($

T^{-1}!

$(-1,3)$, $(1,-3)$, $(3,-1)$, $(1,5)$

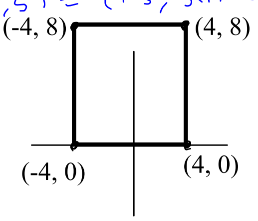
$T^{-1}(x,y) = (u,v) = (x-y,\ 3x+y)$

$T^{-1}(-1,3) = (-1-3,\ 3(-1)+3) = (-4,0)$

$T^{-1}(1,-3) = (1+3,\ 3(1)-3) = (4,0)$

$T^{-1}(3,-1) = \boxed{(3-1,\ 3(3)+1) = (2,10)}$ No. $(3 - (-1), 3(3)+(-1)) = (4, 8)$

$T^{-1}(1,5) = (1-5,\ 3(1)+5) = (-4,8)$



1–6 Find the Jacobian of the transformation.

1. $x = 5u - v, \quad y = u + 3v$ **2.** $x = uv, \quad y = u/v$

3. $x = e^{-r}\sin\theta, \quad y = e^{r}\cos\theta$ **4.** $x = e^{s+t}, \quad y = e^{s-t}$

2 $x = uv$ $y = \frac{u}{v} = uv^{-1}$

$x_u = v$ $y_u = v^{-1} = \frac{1}{v}$

$x_v = u$ $y_v = -uv^{-2} = -\frac{u}{v^2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v & u \\ v^{-1} & -uv^{-2} \end{vmatrix} = \left| -uv^{-1} - uv^{-1} \right|$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \boxed{2\frac{u}{v} = \frac{2u}{v} = 2uv^{-1}}$$

Just the determinant, NOT the absolute value of the determinant!

7–10 Find the image of the set S under the given transformation.

7. $S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\}$;
$x = 2u + 3v, \ y = u - v$

8. S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $\quad x = v, \ y = u(1 + v^2)$

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$x_u = v$

$x_v = u$

$y = \frac{u}{v} = uv^{-1}$

$y_u = v^{-1} = \frac{1}{v}$

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$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v & u \\ v^{-1} & -uv^{-2} \end{vmatrix} = \left| -uv^{-1} - uv^{-1} \right|$$

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19–23 Evaluate the integral by making an appropriate change of variables.

19. $\iint_R \frac{x-2y}{3x-y}\,dA$, where R is the parallelogram enclosed by the lines $x-2y=0$, $x-2y=4$, $3x-y=1$, and $3x-y=8$

20. $\iint_R (x+y)e^{x^2-y^2}\,dA$, where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$

(19)

$x-2y=0$ $\quad y=\frac{1}{2}x$

x	y
0	0

$x-2y=4$

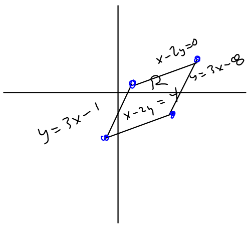
x	y
0	-2
4	0

$3x-y=1$

x	y
0	-1
$\frac{1}{3}$	0

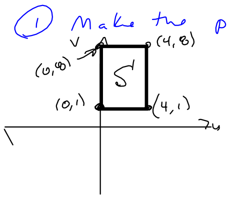
$3x-y=8$

x	y
0	-8
$\frac{8}{3}$	0



2 approaches:

(1) Make the parallelogram a square.



(2) Make a nice function

$$\frac{x-2y}{3x-y} \longrightarrow \frac{u}{v}$$

This requires finding $x(u,v)$ & $y(u,v)$

$u = x-2y$ $\quad T^{-1}: \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}$

$v = 3x-y$

Postulate $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$ $\qquad T$?

$u_x = 1 \quad v_x = 3$

$u_y = -2 \quad v_y = -1$

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = 5 \overset{?}{\Longrightarrow}$$

$|T| = \frac{1}{|T^{-1}|}$ $\qquad$ we know that $\begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

want $x = x(u,v)$, $y = y(u,v)$

$$T^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$T^{-1}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1u+2v \\ -3u+v \end{bmatrix}$$

$\Longrightarrow x_u = -\frac{1}{5}$, $x_v = \frac{2}{5}$ $\Longrightarrow \begin{vmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{vmatrix} = -\frac{1}{25}+\frac{6}{25} = \frac{5}{25} = \frac{1}{5}$

$y_u = -\frac{3}{5}$, $y_v = \frac{1}{5}$

NOTE This is not the same as $\frac{1}{5}\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = 1$

But $\frac{1}{25}\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = \frac{1}{5}$

$$\iint_R \frac{x-2y}{3x-y}\,dA$$

$$= \frac{1}{5}\int_0^4\int_1^8 \frac{u}{v}\,dv\,du = \frac{1}{5}\int_1^8\int_0^4 \frac{u}{v}\,du\,dv = \frac{1}{5}\int_1^8 \left[\frac{1}{2}\frac{u^2}{v}\right]_0^4 dv$$

$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

$$= \frac{1}{5}\int_1^8 \frac{8}{v}\,dv = \frac{8}{5}\ln|v|\Big]_1^8 = \frac{8}{5}\left[\ln(8)-\ln(1)\right]$$

$$= \boxed{\frac{8}{5}\ln(8)}$$

Nov 13-2:00 PM