

The **spherical coordinates**  $(\rho, \theta, \phi)$  of a point  $P$  in space are shown in Figure 1, where  $\rho = |OP|$  is the distance from the origin to  $P$ ,  $\theta$  is the same angle as in cylindrical coordinates, and  $\phi$  is the angle between the positive  $z$ -axis and the line segment  $OP$ . Note that

$$\rho \geq 0 \quad 0 \leq \phi \leq \pi$$

$$\sqrt{x^2 + y^2} \leq r$$

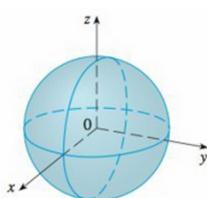
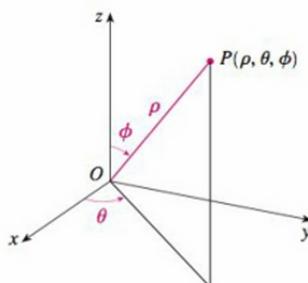


FIGURE 2  $\rho=c$ , a sphere

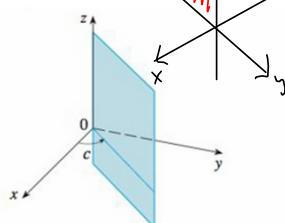
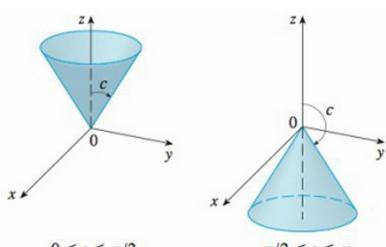


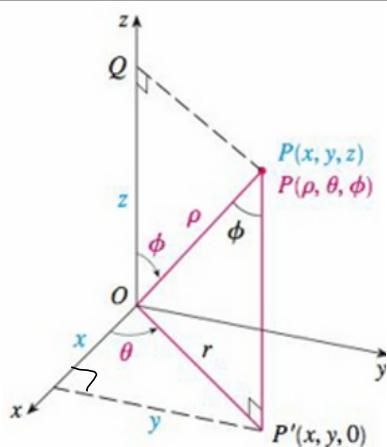
FIGURE 3  $\theta=c$ , a half-plane

$$\theta = \frac{3\pi}{2}$$



$$0 < c < \pi/2$$

FIGURE 4  $\phi=c$ , a half-cone



$$z = \rho \cos \phi \quad r = \rho \sin \phi$$

$$\frac{z}{\rho} = \cos \phi \quad \frac{r}{\rho} = \sin \theta$$

$$z = \rho \cos \theta$$

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

$$\frac{x}{r} = \cos \theta \quad \frac{y}{r} = \sin \theta$$

$$x = r \cos \theta \quad y = r \sin \theta$$

$$= \rho \sin \phi \cos \theta \quad = \rho \sin \phi \sin \theta$$

FIGURE 5

$$\rho^2 = x^2 + y^2 + z^2$$

$\rho = \sqrt{x^2 + y^2 + z^2}$  by distance formula in 3-D.

1-2 Plot the point whose spherical coordinates are given. Then find the rectangular coordinates of the point.

2. (a)  $(5, \pi, \pi/2)$  (b)  $(4, 3\pi/4, \pi/3)$

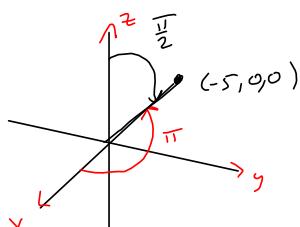
$$(a) (5, \pi, \frac{\pi}{2}) = (\rho, \theta, \phi)$$

$$\begin{aligned} x &= \rho \sin \phi \cos \theta \\ &= 5 \sin \frac{\pi}{2} \cos \pi \\ &= 5(1)(-1) \end{aligned}$$

$$\begin{aligned} y &= \rho \sin \phi \sin \theta \\ &= 5 \left(\sin \frac{\pi}{2}\right) (\sin \pi) \\ &= 5(1)(0) \end{aligned}$$

$$(-5, 0, 0) = (x, y, z)$$

$$\begin{aligned} z &= \rho \cos \phi \\ &= 5 \cos \frac{\pi}{2} = 0 \end{aligned}$$

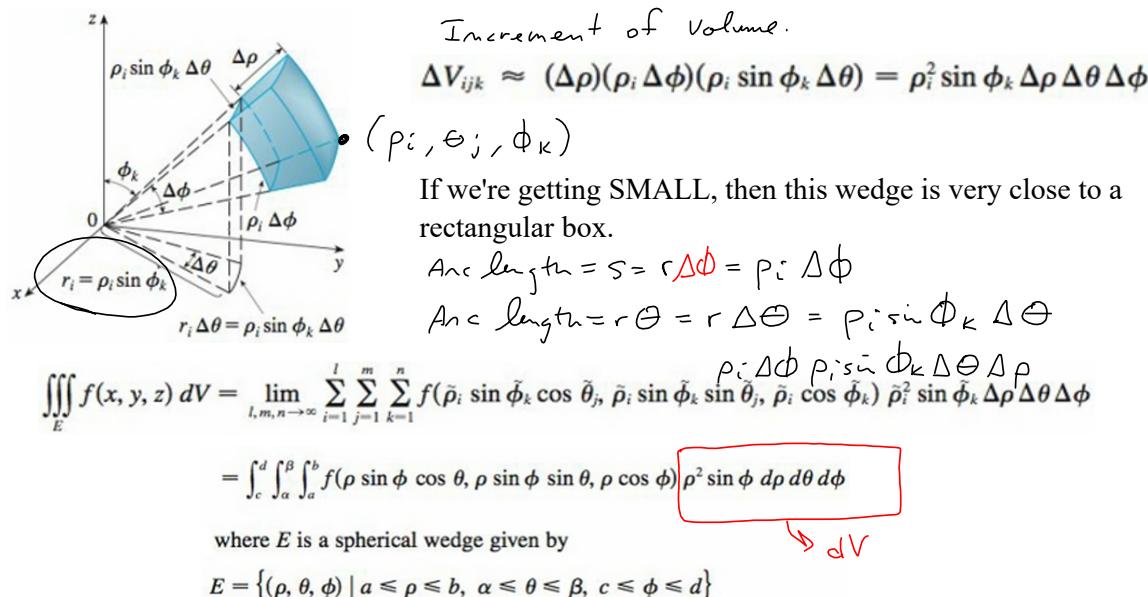
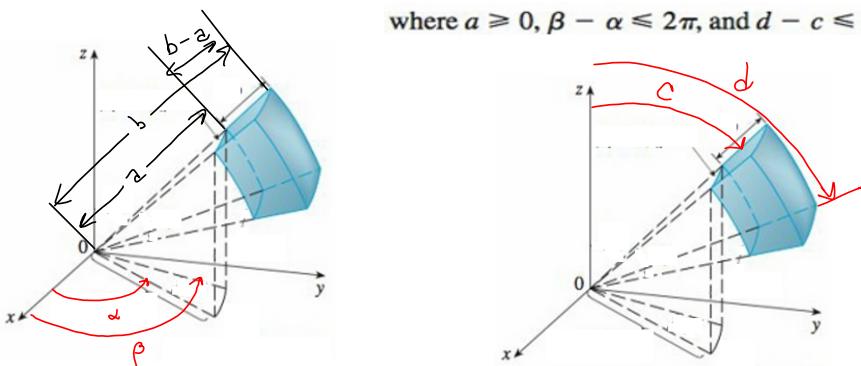


### EVALUATING TRIPLE INTEGRALS WITH SPHERICAL COORDINATES

In the spherical coordinate system the counterpart of a rectangular box is a **spherical wedge** = spherical "rectangular box."

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

where  $a \geq 0, \beta - \alpha \leq 2\pi$ , and  $d - c \leq \pi$ .



Evaluate  $\iiint_B e^{(x^2+y^2+z^2)^{1/2}} dV$ , where  $B$  is the unit ball:

Example  $B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$

$$\int_0^1 \int_0^{2\pi} \int_0^\pi e^{(\rho^2)^{1/2}} \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho$$

Not fun in rectangular coordinates:

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{1/2}} dz \, dy \, dx$$

$$u = \rho^3 \Rightarrow du = 3\rho^2 d\rho$$

$$= \int_0^1 \int_0^{2\pi} \int_0^\pi e^{\rho^3} \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho$$

$$= \frac{1}{3} \int_0^1 e^{\rho^3} \cdot 3\rho^2 d\rho \int_0^{2\pi} \int_0^\pi \sin \phi \, d\phi$$

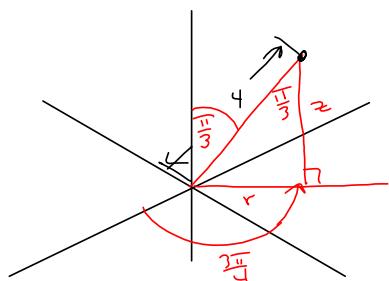
$$\frac{1}{3} \left[ e^{\rho^3} \right]_0^1 \left[ \theta \right]_0^{2\pi} \left[ -\cos \phi \right]_0^\pi = \frac{1}{3} \left[ e^1 - e^0 \right] \left[ 2\pi \right] \left[ -\cos \pi - (-\cos 0) \right]$$

$$= \frac{1}{3} [e - 1] (2\pi) [1 - (-1)] = \frac{4\pi}{3} [e - 1]$$

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi} \exp(\rho^3) \cdot \rho^2 \cdot \sin(\phi) \, d\phi \, d\theta \, d\rho = 4\pi \left( -\frac{1}{3} + \frac{e}{3} \right) = \frac{4\pi}{3} (e - 1)$$

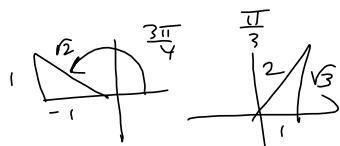
**1-2** Plot the point whose spherical coordinates are given. Then find the rectangular coordinates of the point.

2. (a)  $(5, \pi, \pi/2)$



(b)  $(4, 3\pi/4, \pi/3)$

$$\rho = 4, \theta = \frac{3\pi}{4}, \phi = \frac{\pi}{3}$$



$$x = r \cos \theta = \rho \sin \phi \cos \theta = 4 \sin \frac{\pi}{3} \cos \frac{3\pi}{4}$$

$$= 4 \cdot \frac{\sqrt{3}}{2} \cdot \left(-\frac{1}{\sqrt{2}}\right) = -\frac{4\sqrt{3}}{2\sqrt{2}} = -\frac{4\sqrt{6}}{4} = -\sqrt{6} = -x$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta = 4 \sin \frac{\pi}{3} \sin \frac{3\pi}{4} = \sqrt{6} = y$$

$$z = \cos \phi = \rho \cos \phi = 4 \cos \frac{\pi}{3} = 4 \cdot \frac{1}{2} = 2 = z$$

$$(x, y, z) = (-\sqrt{6}, \sqrt{6}, 2)$$

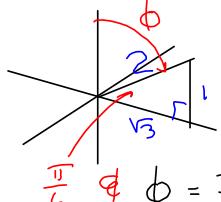
**3-4** Change from rectangular to spherical coordinates.

4. (a)  $(0, \sqrt{3}, 1)$

$$4a) \rho = \sqrt{x^2 + y^2 + z^2}$$

$$= \sqrt{0^2 + \sqrt{3}^2 + 1^2}$$

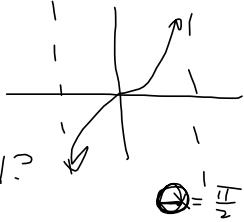
$$= 2$$



(b)  $(-1, 1, \sqrt{6})$

$$\theta = \frac{\pi}{2} \text{ by p.c.}$$

In general,  
 $\tan \theta = \frac{y}{x} = \frac{\sqrt{3}}{0}$  undefined?



$$\phi = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3} = \phi$$

$$(\rho, \theta, \phi) = (2, \frac{\pi}{2}, \frac{\pi}{3})$$

**5-6** Describe in words the surface whose equation is given.

6.  $\rho = 3$

Sphere of  
radius 3.

**7-8** Identify the surface whose equation is given.

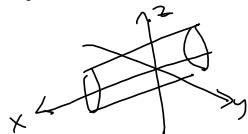
**8.**  $\rho^2(\sin^2\phi \sin^2\theta + \cos^2\phi) = 9$

$$z = \rho \cos \phi \quad r = \rho \sin \phi \quad \rho^2 = x^2 + y^2 + z^2$$

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

$$\Rightarrow \rho^2 \sin^2\phi \sin^2\theta + \rho^2 \cos^2\phi = r^2 \sin^2\theta + z^2 = y^2 + z^2 = 9$$

Circular cylinders of radius  $r=3$ , centered on the  $x$ -axis

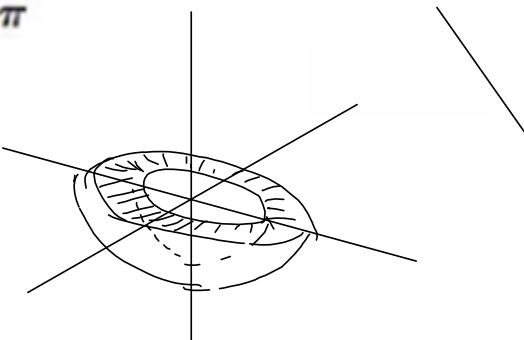


**11-14** Sketch the solid described by the given inequalities.

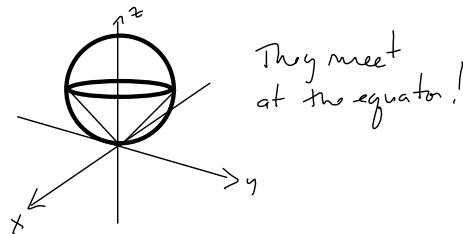
**12.**  $2 \leq \rho \leq 3, \quad \pi/2 \leq \phi \leq \pi$

**14.**  $\rho \leq 2, \quad \rho \leq \csc \phi$

Bowl!



15. A solid lies above the cone  $z = \sqrt{x^2 + y^2}$  and below the sphere  $x^2 + y^2 + z^2 = z$ . Write a description of the solid in terms of inequalities involving spherical coordinates.



$$z = \sqrt{x^2 + y^2} \quad \text{in } yz\text{-plane}$$

$$z = \sqrt{0^2 + y^2} = \sqrt{y^2} = |y|$$

$$z = |y| \Rightarrow$$

$$z = y \text{ or } z = -y$$

Where does the cone intersect the sphere?

$$z = \sqrt{x^2 + y^2} = x^2 + y^2 + z^2$$

in the  $yz$ -plane:

$$z = |y| = y^2 + z^2$$

In the  $yz$ -plane,  $y = z$

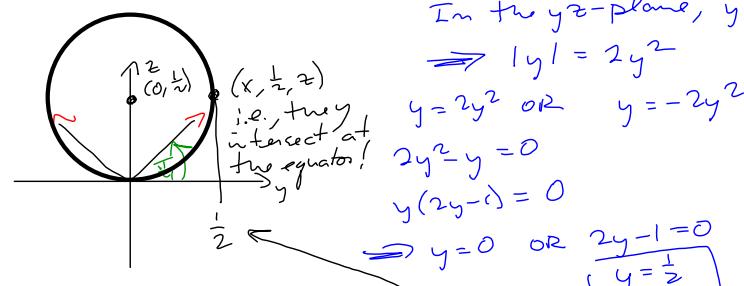
$$\Rightarrow |y| = 2y^2$$

$$y = 2y^2 \text{ or } y = -2y^2$$

$$2y^2 - y = 0$$

$$y(2y - 1) = 0$$

$$\Rightarrow y = 0 \text{ or } \frac{2y - 1}{y} = 0 \boxed{y = \frac{1}{2}}$$



$$z = x^2 + y^2 + z^2$$

$$x^2 + y^2 + z^2 - z + \left(\frac{1}{2}\right)^2 = 0 + \frac{1}{4}$$

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$$r = \frac{1}{2}, \text{ center} = (0, 0, \frac{1}{2})$$

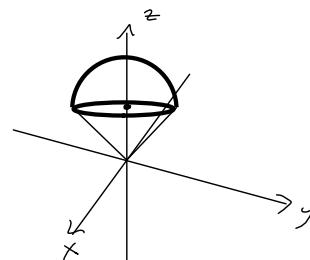
Above the cone:

$$\rho \cos \phi = z = \sqrt{x^2 + y^2} = r = \rho \sin \phi$$

$$\cos \phi = \sin \phi$$

$$\frac{\sin \phi}{\cos \phi} = 1 \Rightarrow \phi = \frac{\pi}{4}$$

$$\boxed{0 \leq \phi \leq \frac{\pi}{4}} \text{ keeps it above the cone.}$$



Below the sphere:

$$z = x^2 + y^2 + z^2 = z$$

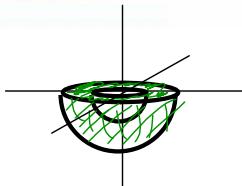
$$\rho^2 = \rho \cos \phi$$

$$\rho = \cos \phi$$

$$\boxed{0 \leq \rho \leq \cos \phi}$$

- 17-18 Sketch the solid whose volume is given by the integral and evaluate the integral.

$$18. \int_0^{2\pi} \int_{\pi/2}^{\pi} \int_1^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

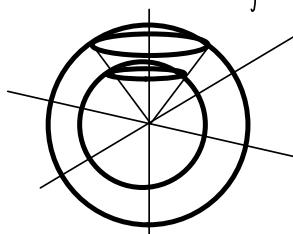
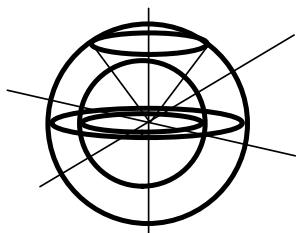


26. Evaluate  $\iiint_E xyz \, dV$ , where  $E$  lies between the spheres  $\rho = 2$  and  $\rho = 4$  and above the cone  $\phi = \pi/3$ .

$$dV =$$

$$d\rho \, d\theta \, d\phi$$

$$\begin{aligned}x &= r \cos \theta = \rho \sin \phi \cos \theta \\y &= r \sin \theta = \rho \sin \phi \sin \theta \\z &= \rho \cos \phi\end{aligned}$$



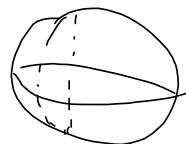
$$\begin{aligned}&\int_0^{\frac{\pi}{3}} \int_0^{2\pi} \int_2^4 \cancel{\rho \sin \phi \cos \theta} \cancel{\rho \sin \phi \sin \theta} \cancel{\rho \cos \phi} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \quad ?! \\&= \int_0^{\frac{\pi}{3}} \sin^3 \phi \cos \phi \, d\phi \int_0^{2\pi} \sin \theta \cos \theta \, d\theta \int_2^4 \rho^5 \, d\rho = \left[ \frac{\sin^4 \phi}{4} \right]_0^{\frac{\pi}{3}} \left[ \frac{\sin^2 \theta}{2} \right]_0^{2\pi} \left[ \frac{\rho^6}{6} \right]_2^4 \\&= \frac{4^6 - 2^6}{6} \\&= 2^6(2^6 - 1) \\&= 64(63) \\&= \frac{164}{63} \\&= \frac{3840}{4032}\end{aligned}$$

**35-38** Use cylindrical or spherical coordinates, whichever seems more appropriate.

- 36.** Find the volume of the smaller wedge cut from a sphere of radius  $a$  by two planes that intersect along a diameter at an angle of  $\pi/6$ .

$$x^2 + y^2 + z^2 = a^2$$

$$\theta = 0 \quad \theta = \frac{\pi}{6}$$



$$0 \leq \theta \leq \frac{\pi}{6}$$

$$0 \leq \rho \leq a$$

$$0 \leq \phi \leq \pi$$

$$\int_0^{\frac{\pi}{6}} \int_0^a \int_0^\pi \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} d\theta \int_0^a \rho^2 d\rho \int_0^\pi \sin \phi \, d\phi = \left( \frac{\pi}{6} \right) \left( \frac{a^3}{3} \right) \left( -\cos \phi \right)_0^\pi$$

$$= \frac{\pi a^3}{18} (-\cos \pi - (-\cos 0))$$

$$= \frac{\pi a^3}{18} (1 + 1) = \boxed{\frac{\pi a^3}{9}} = \text{Volume}$$