15.7 - Triple Integrals in Cylindrical Coordinates

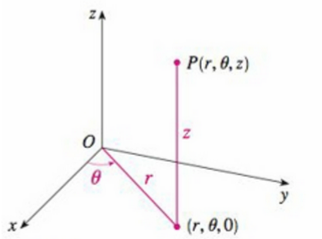


FIGURE 2
The cylindrical coordinates of a point

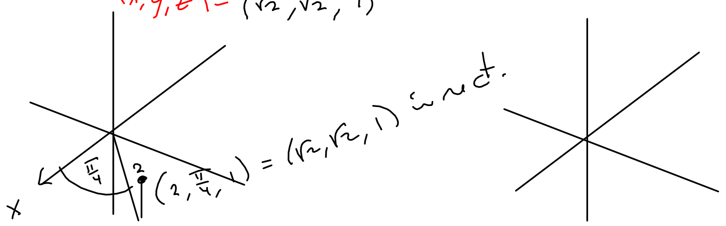
$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

1–2 Plot the point whose cylindrical coordinates are given. Then find the rectangular coordinates of the point.

1. (a) $(2, \pi/4, 1)$ (b) $(4, -\pi/3, 5)$

$(x,y,z) = (\sqrt{2}, \sqrt{2}, 1)$



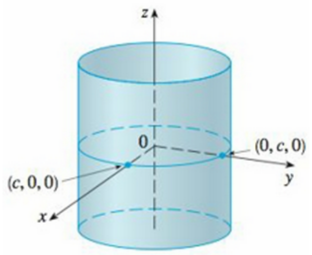
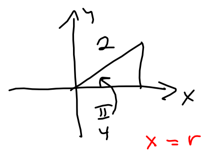
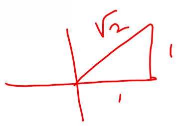


FIGURE 4
$r = c$, a cylinder





$x = r\cos\theta$
$= 2\cos\frac{\pi}{4} = 2\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

7–8 **Identify the surface whose equation is given.**

7. $z = 4 - r^2$ $= 4 - (x^2+y^2)$ = Paraboloid

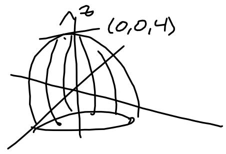
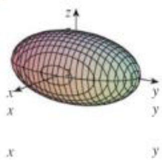
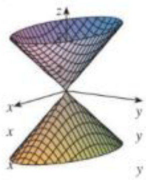
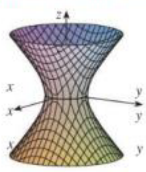
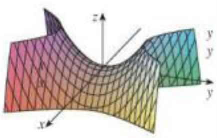
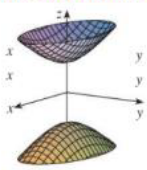


Table 1 Graphs of Quadric Surfaces S12.6?

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

EVALUATING TRIPLE INTEGRALS WITH CYLINDRICAL COORDINATES

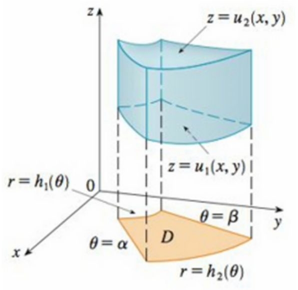


FIGURE 6

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is given in polar coordinates by TYPE 1, Always.

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

$G(x,y) = G(r\cos\theta, r\sin\theta)$

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

$\to r\,dr\,d\theta$

Recall from Section 15.4 on Double Integrals in Polar Coordinates:

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \boxed{f(r\cos\theta, r\sin\theta)}\, r\, dr\, d\theta$$

$\int_{u_1(x,y)}^{u_2(x,y)} \sim dz = G(x,y)$ OR $G(r\cos\theta, r\sin\theta)$

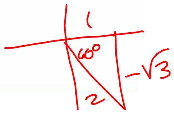
Combine to obtain the triple integral in CYLINDRICAL coordinate.

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

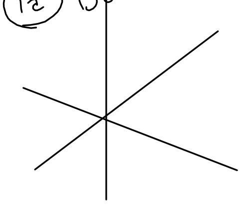
1–2 Plot the point whose cylindrical coordinates are given. Then find the rectangular coordinates of the point.

1. (a) $(2, \pi/4, 1)$ (b) $(4, -\pi/3, 5)$

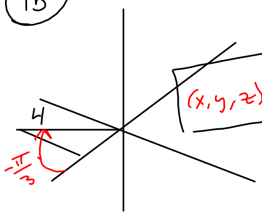
2. (a) $(1, \pi, e)$ (b) $(1, 3\pi/2, 2)$



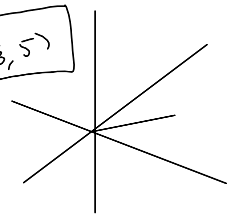
1a Done

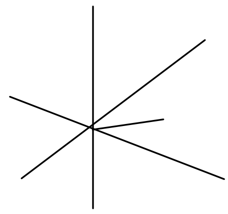


1b



$\boxed{(x,y,z) = (2, -2\sqrt{3}, 5)}$





$x = 4\cos\left(-\frac{\pi}{3}\right)$
$= 4 \cdot \frac{1}{2} = 2$
$y = 4\sin\left(-\frac{\pi}{3}\right) = 4 \cdot \left(-\frac{\sqrt{3}}{2}\right)$
$= -2\sqrt{3}$

3–4 Change from rectangular to cylindrical coordinates.

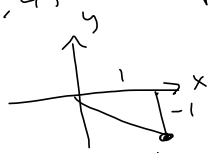
3. (a) $(1, -1, 4)$ (b) $(-1, -\sqrt{3}, 2)$

4. (a) $(2\sqrt{3}, 2, -1)$ (b) $(4, -3, 2)$

3a

$(1, -1, 4) \Rightarrow r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

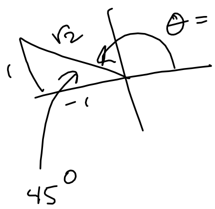
$\theta = \arctan\left(-\frac{1}{1}\right)$
$= -\frac{\pi}{4} = -45°$



$\arctan(-1) = -45°$,
but
this ain't $-45°$.

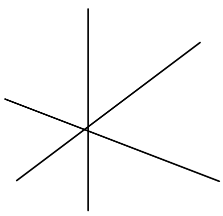
Do draw picture

$\boxed{(\sqrt{2}, -\frac{\pi}{4}, 4) = (r, \theta, z)}$



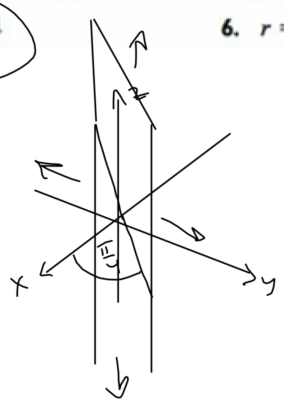
$\theta = 135° = \frac{3\pi}{4}$

But arctan(-1) tells you QIV, but you know it's QII. Draw the picture. Interpret.



5–6 Describe in words the surface whose equation is given.

5. $\theta = \pi/4$ **6.** $r = 5$



Vertical Plane $\boxed{y = x}$

7–8 Identify the surface whose equation is given.

7. $z = 4 - r^2$ **8.** $2r^2 + z^2 = 1$

#7 Done already.
See video 00-general remarks.

9–10 Write the equations in cylindrical coordinates.

9. (a) $z = x^2 + y^2$ (b) $x^2 + y^2 = 2y$

10. (a) $3x + 2y + z = 6$ (b) $-x^2 - y^2 + z^2 = 1$

10a $\boxed{3(r\cos\theta) + 2(r\sin\theta) + z = 6}$

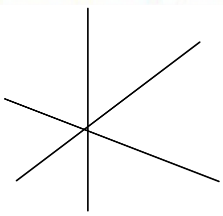
$3r\cos\theta + 2r\sin\theta - 6 = z$

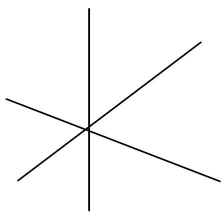
10b $-x^2 - y^2 + z^2 = 1 \Rightarrow -(x^2+y^2) + z^2 = \boxed{-r^2 + z^2 = 1}$

11–12 Sketch the solid described by the given inequalities.

11. $0 \leq r \leq 2, \quad -\pi/2 \leq \theta \leq \pi/2, \quad 0 \leq z \leq 1$

12. $0 \leq \theta \leq \pi/2, \quad r \leq z \leq 2$





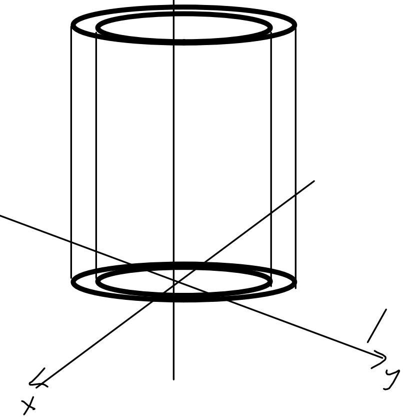
13. A cylindrical shell is 20 cm long, with inner radius 6 cm and outer radius 7 cm. Write inequalities that describe the shell in an appropriate coordinate system. Explain how you have positioned the coordinate system with respect to the shell.

14. Use a graphing device to draw the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 5 - x^2 - y^2$.

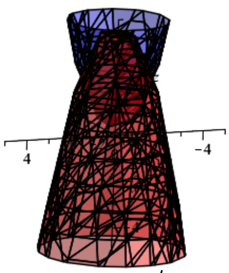
15–16 Sketch the solid whose volume is given by the integral and evaluate the integral.

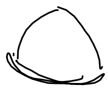
15. $\int_0^4 \int_0^{2\pi} \int_r^4 r\, dz\, d\theta\, dr$

$0 \le z \le 20,\quad 6 \le r \le 7$



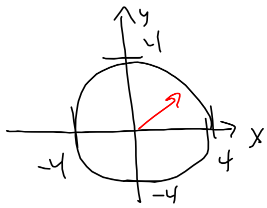
14

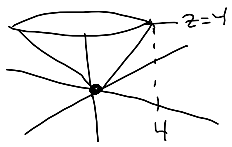




15 $\int_0^4 \int_0^{2\pi} \int_r^4 r\, dz\, d\theta\, dr$

$z = r$ to $z = 4$ $\quad z = \sqrt{x^2+y^2}$





$$\int_0^4 \int_0^{2\pi} \Big[r \cdot z\Big]_r^4 d\theta\, dr$$

$$= \int_0^4 \int_0^{2\pi} (4r - r^2)\, d\theta\, dr$$

$$= \int_0^4 (4r - r^2)\Big[\theta\Big]_0^{2\pi} dr = 2\pi \int_0^4 (4r - r^2)\, dr = 2\pi\left[2r^2 - \frac{r^3}{3}\right]_0^4$$

$$= 2(4)^2 - \frac{4^3}{3} = 32 - \frac{64}{3} = \frac{96 - 64}{3} = \boxed{\frac{32}{3}}$$

18. Evaluate $\iiint_E (x^3 + xy^2)\, dV$, where E is the solid in the first octant that lies beneath the paraboloid $z = 1 - x^2 - y^2$.

20. Evaluate $\iiint_E x\, dV$, where E is enclosed by the planes $z = 0$ and $z = x + y + 5$ and by the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

27–28 Evaluate the integral by changing to cylindrical coordinates.

27. $\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} xz \, dz \, dx \, dy$